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HISTORY

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Cf. Vygodskiĭ, same Trudy, no. 1, 217–295 (1948); these Rev. **11**, 150.

Segre, Beniamino. Obituary: Ettore Bortolotti. Rend. Sess. Accad. Sci. Ist. Bologna (N.S.) **52** (1947/48), 47–86 (1949).

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Gnedenko, B. V. On the works of N. I. Lobačevskiĭ in the theory of probability. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **2**, 129–136 (1949). (Russian)

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Hil′kevič, E. K. From the history of the dissemination and development of the ideas of N. I. Lobačevskiĭ in the years 60–70 of the 19th century. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **2**, 168–230 (1949). (Russian)

Popovici, Constantin. The third centenary of the birth of Newton. Isaac Newton and modern problems of celestial mechanics. Acad. Şti. România. Ser. III. Mem. Monogr. No. 6, 24 pp. (undated).
Translation of a lecture given in Bucarest in January, 1943, and published in a Romanian text during that year.

Szökefalvi-Nagy, Béla. The mathematical oeuvre of F. Riesz. Mat. Lapok **1**, 170–182 (1950). (Hungarian. Russian and English summaries)

***Leggett, H. W. Bertrand Russell, O. M.** Philosophical Library, New York, N. Y., 1950. 79 pp. $3.75.
This is a pleasantly edited and nicely illustrated account of the life and works of Bertrand Russell. Chapter V of the book (pp. 52–64) on "Analytical Empiricism" deals with Russell's views on mathematics and logic, and though the author qualifies it as "a rather crude simplification of some of his [Russell's] ideas," it may be helpful for a first orientation. The last chapter of the book contains a list of Russell's principal works from 1896 to 1949, with some comment. *D. J. Struik* (Cambridge, Mass.).

Obituary: D. M. Sincov. Zapiski Naučno-Issled. Inst. Mat. Meh. Har′kov. Mat. Obšč. (4) **19**, 5–19 (1 plate) (1948). (Russian)

Allen, R. G. D. The work of Eugen Slutsky. Econometrica **18**, 209–216 (1 plate) (1950).

Bašmakova, I. G. The foundations of the theory of divisibility in the works of E. I. Zolotarev. Trudy Sem. MGU Istor. Mat. Istor.-Mat. Issledov. no. **2**, 231–351 (1949). (Russian)

FOUNDATIONS

Dürr, Karl. Der Begriff der Funktion in der symbolischen Logik. Synthèse 7, 418–427 (1949).

The paper is a historical review of the concept of function in the logical work of Schröder, Frege, Peano, and Russell.

D. Nelson (Washington, D. C.).

Marcus, Ruth Barcan. The elimination of contextually defined predicates in a modal system. J. Symbolic Logic 15, 92 (1950).

It was pointed out by G. Bergmann [same J. 13, 140 (1948)] that in some nonextensional languages contextually defined predicates are not always eliminable. The author of the present note, on the other hand, shows by an example that there exist nonextensional languages in which such predicates are always eliminable.

J. C. C. McKinsey (Santa Monica, Calif.).

Kustaanheimo, Paul. Über die Vollständigkeit der Axiomensysteme mit einem endlichen Individuenbereich. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 63, 44 pp. (1949).

The author is concerned with the problem of establishing necessary and sufficient conditions that a formal axiom system be complete. He finds such conditions for the case that the axiom system involves only one-place predicates and also for the case of axiom systems involving many-place predicates, but under the assumption that the universe of discourse contains a fixed finite number of elements. He treats separately the cases of axiom systems that can be formalized in the restricted predicate calculus, and axiom systems that can be formalized in the second-order predicate calculus. The results appear to be correct, though not surprising.

J. C. C. McKinsey (Santa Monica, Calif.).

Mostowski, Andrzej. An undecidable arithmetical statement. Fund. Math. 36, 143–164 (1949).

A system S of axioms is given, equivalent to those of Bernays and Gödel for set theory, and containing but one primitive symbol ε. Let $\phi(\varepsilon)$ denote the conjunction of the axioms of S, preceded by general quantifiers for all free variables. Consider the formula

$$\Delta(\varepsilon) \equiv (Ex)(\phi_{[x]}(\varepsilon) \cdot (y)(y\varepsilon[x] \rightarrow y \subset [x])).$$

Here $[x]$ is defined by $y\varepsilon[x] \equiv y\varepsilon x \cdot (Et)(t\varepsilon x \cdot (y\varepsilon t \vee t\varepsilon y))$; ϕ_u is the formula which results when in ϕ every general quantifier (z) is replaced by $(z)(z\varepsilon u \rightarrow \cdots)$ and every existential quantifier (Ez) by $(Ez)(z\varepsilon u \cdot \cdots)$. The author proves: (i) There is a set a such that $\phi_{[a]}(\varepsilon) \cdot \Delta_{[a]}(\varepsilon)$; (ii) there is a set b such that $\phi_{[b]}(\varepsilon) \cdot \Delta_{[b]}(\varepsilon)$. From (ii) it follows that $\Delta(\varepsilon)$ cannot be demonstrable in S, whereas from (i) this follows for $\overline{\Delta(\varepsilon)}$. The proof of (i) and (ii) rests on set theory as symbolized in S, with an additional axiom F ensuring the existence of at least one inaccessible aleph. In order to obtain an undecidable arithmetical statement, the theorem of Skolem-Löwenheim is used to prove that for every set-theoretical formula $\psi(\varepsilon)$ there is an elementary arithmetical formula $\mathfrak{G}(A, B)$ such that $(Ex)(\psi_{[x]}(\varepsilon) \cdot (y)(y\varepsilon[x] \rightarrow y \subset [x]))$ is equivalent to $(EA)(A \subset N \cdot (B)(B \subset N \rightarrow \mathfrak{G}(A, B)))$, where N is a natural number. The formula $\mathfrak{F}$ resulting in this way from $\Delta(\varepsilon)$ is undecidable in S, but follows from S after adjunction of F.

A. Heyting (Amsterdam).

Fitch, Frederic B. A demonstrably consistent mathematics. I. J. Symbolic Logic 15, 17–24 (1950).

The author develops his consistent system K' in a way slightly different from that of his earlier papers [J. Symbolic Logic 13, 95–106 (1948); 14, 9–15 (1949); these Rev. 9, 559; 10, 669]. Attention is focussed here as previously upon the foundations of the theories of rational and real numbers. Rationals are here regarded as ordered triples of natural numbers, rather than as ordered pairs, and hence signed rationals may readily be dealt with. If $[a, b, c]$ is a triple of the appropriate kind, roughly speaking, it may be regarded as the rational number $(a-b)/c$. The theory of real numbers can then be developed mutatis mutandis as in the previous papers, but here obviously the nonnegative reals can be accommodated as well.

R. M. Martin.

Fitch, Frederic B. A further consistent extension of basic logic. J. Symbolic Logic 14, 209–218 (1950).

This paper is concerned with the author's foundation for a consistent theory of real numbers. The consistent system K [same J. 7, 105–114 (1942); these Rev. 4, 125], it will be recalled, is a "basic logic" according to the author in the sense that every recursively enumerable subclass of its (well-formed) expressions is represented in K. The consistent system K' [same J. 13, 95–106 (1948); 14, 9–15 (1949); these Rev. 9, 559; 10, 669] is an extension of K providing for ways of dealing with negation and universal quantification.

In the paper under review a further consistent extension of K is considered, one in which a symbol for a certain kind of conditionality plays a role as well as a symbol for the modal concept of necessity. This system, called L, permits the introduction of various kinds of restricted universal quantifiers which are not available in K'. It is shown that the system L is consistent in the same sense in which K' is.

R. M. Martin (Philadelphia, [illegible]).

Germansky, Baruch. A new set of axioms sufficient for the development of the theory of natural numbers. Riveon Lematematika 3, 65–67 (1949). (Hebrew. English summary)

Gégalkine, I. Sur le problème de la résolubilité pour les classes finies. Učenye Zapiski Moskov. Gos. Univ. 100, Matematika, Tom I, 155–211 (1946). (Russian. French summary)

This paper concerns the conditions under which the expression

$$H(F, f_1, \cdots, f_r) = (Ex)(y)F(x, y) \vee (Ex_1) \cdots (Ex_s)B(F, f_1, \cdots, f_r; x_1, \cdots, x_s),$$

where $F, f_1, \cdots, f_r$ are functions of two arguments, is identically true on all finite sets (identical in the finite). Ackermann [Math. Ann. 112, 419–432 (1936)] proved for the case $r=0$: If H is identical in a set of $s+1$ elements, then H is identical in the finite. The author [Rec. Math. [Mat. Sbornik] N.S. 6(48), 185–198 (1939); these Rev. 1, 322] proved that it suffices that H is true for a set U of special functions F. In this paper the latter result is extended to the general case. The method of proof is the following. A function $F(x, y)$ on a set S_n of n elements $c_1, \cdots, c_n$ is represented by the matrix of its values $(F(c_i, c_k)) = P$, truth being written as 1, falsehood as 0. To a subclass of S_n corresponds

a (diagonal) minor of P. By a permutation in S_n, P is converted into a "similar" matrix. The set U is composed of the matrices $\sigma_2, \cdots, \sigma_n, \tau_n$; σ_n and τ_n have 1 at the intersection of the kth column and the $(k+1)$th row, $k=1, \cdots, n-1$; in σ_n also the last element of the first row is 1, all other elements are 0. If P contains a zero column, H is true in P. If a diagonal element in P is 1, the question can be reduced to one in the calculus of propositions. Suppose now that H satisfies the matrices of U. Let P be a matrix which has no zero column, while all its diagonal elements are 0. It is shown that such a matrix has a minor similar to some σ_k. If $k \leqq s$, H satisfies σ_k; consequently, H satisfies P. If $k>s$, τ_s is a minor of σ_k. Since H satisfies τ_s, H satisfies P. In this reasoning $f_1, \cdots, f_r$ are left arbitrary. A shorter decision method, based on the disjunctive normal form of B, is given for the case $r=0$. *A. Heyting.*

Griss, G. F. C. Negationless intuitionistic mathematics. II. Nederl. Akad. Wetensch., Proc. **53**, 456–463 = Indagationes Math. **12**, 108–115 (1950).

The author briefly restates his position concerning negation and the introduction of unsatisfiable predicates in mathematics and continues his development of an earlier paper [same Proc. **49**, 1127–1133 = Indagationes Math. **8**, 675–681 (1946); these Rev. **8**, 307] noting certain revisions to that work. Here he discusses fundamental properties of integers and rational numbers. *D. Nelson.*

Schmidt, Arnold. Wie dürfen wir mit dem Unendlichen umgehen? (Die Grundlage des mathematischen Intuitionismus). Math.-Phys. Semesterber. **1**, 200–212 (1950).

The author presents a brief general introduction to intuitionism. He outlines arguments of the Brouwerian type for the unacceptability of such classical items as the principle of excluded middle in logic, the fundamental theorem of algebra, and the convergence of a bounded monotonic sequence. He also discusses Brouwer's theorem asserting the continuity of a function over any interval on which it is defined. *D. Nelson* (Washington, D. C.).

Gallego Diaz, J. A universal metric for the experimental sciences. Gaz. Mat., Lisboa **10**, no. 41–42, 1–4 (1949). (Spanish)

The author asserts that a better description of certain physical phenomena (capable of being represented in a coordinate space of two or more dimensions) would be obtained by using a metric invariant under a change of scale $(x'=ax, y'=by)$ rather than the Euclidean metric whose characteristic property of invariance under rotation of axes is usually without significance in the representation of such phenomena. Assuming, therefore, that (1) distance AB of points $A(x_1, y_1)$, $B(x_2, y_2)$ is a function $F(x_1/x_2, y_1/y_2)$, (2) $AB=BA$, and (3) $AA=0$, one is led to the metric $AB=[\ln(x_1/x_2)\cdot\ln(y_1/y_2)]^k$, or (in differential form) $ds^2=(dx/x)(dy/y)$. With this metric it is found that the curves $y=C\cdot x^m$ are geodesics. [Reviewer's note. Restricting the space to the interior of the first quadrant, the curves $y=C\cdot x^m$, $C>0$, are straight lines.] Extremals for the integrals $\int x^m x^n (dxdy)^k$ and $\int ds/v(x, y)$, $v(x, y)=k[(a-y)/x]^k$ (analogues of Hamilton's principle and the brachistochrone integral) are found. The paper concludes with a brief application to biology and biophysics. *L. M. Blumenthal.*

Frank, Philipp. Metaphysical interpretations of science. I. British J. Philos. Sci. **1**, 60–74 (1950).

Frank, Philipp. Metaphysical interpretations of science. II. British J. Philos. Sci. **1**, 77–91 (1950).

ALGEBRA

Goodwin, E. T. Note on the evaluation of complex determinants. Proc. Cambridge Philos. Soc. **46**, 450–452 (1950).

Let A be a square matrix of order n with complex elements. The author shows that $\det A$ can be evaluated without the use of any operations involving complex numbers. Let A_1 be the real square matrix of order $2n$ obtained by replacing each element of A by its matrix equivalent. Express A_1, by means of the triangular resolution, as the product B_2C_2 of a unit lower triangular matrix B_2 and an upper triangular matrix C_2. Then $\det A$ can be evaluated in terms of the elements of B_2 and C_2.

G. B. Price (Lawrence, Kan.).

Parodi, M. Sur une propriété des déterminants gauches. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 81–82 (1949).

It is shown that all the roots of

$$F(z)=\begin{vmatrix} a+z & a_{12} & \cdots & a_{1n} \\ -a_{12} & a+z & \cdots & a_{2n} \\ & \cdots & \cdots & \\ -a_{1n} & -a_{2n} & \cdots & a+z \end{vmatrix}=0 \qquad (a_{ij} \text{ real}, a>0),$$

are situated upon the line $x=-a$. If, moreover, m is a complex number with positive real part, then every zero of $F(z)+mF'(z)$ has its real part less than $-a$ provided that the roots of $F(z)=0$ are all simple. *L. M. Blumenthal.*

Stoyakovitch, M. Sur les determinants semi-adjoints. Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 137–144 (1949). (Serbian. French summary)

The "semi-adjoint" of $\|a_{ij}\|$ is obtained when the elements of certain rows are replaced by their minors.

W. Feller (Princeton, N. J.).

Mitrinovitch, D. S. Sur un déterminant du type d'Escherich. Fac. Philos. Skopje. Sect. Sci. Nat. Annuaire **2**, 137–140 (1949). (Serbian. Russian and French summaries)

It is proved that

$$\begin{vmatrix} a^n & -a^{n-1}b & a^{n-2}b^2 & \cdots & (-1)^{n-1}ab^{n-1} & (-1)^n b^n \\ n & 1 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & n-2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & n \end{vmatrix} = n!(a+b)^n.$$

A. W. Goodman (Lexington, Ky.).

Popov, B. S. Remarques sur quelques determinants. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2**, 197–205 (1949). (Macedonian. Russian and French summaries)

The author points out that the value

$$V_n=a_0a_1\cdots a_n\sum_{r=0}^{n}(a_r^{-1}+xr^2\sum_{k=0}^{n-r}(a_ka_{k+1})^{-1})$$

given by Cesàro [Mathesis 2, 5–10 (1892)] for the determinant

$$D=\begin{vmatrix} x+a_0+a_1 & x-a_1 & 0 & \cdots & 0 \\ x-a_1 & x+a_1+a_2 & x-a_2 & \cdots & 0 \\ 0 & x-a_2 & x+a_2+a_3 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & x+a_{n-1}+a_n \end{vmatrix}$$

is incorrect, since V_n is linear in x, while D_n satisfies the recurrence relation $D_k=(x+a_{k-1}+a_k)D_{k-1}-(x-a_{k-1})^2D_{k-2}$ and hence cannot be linear for three successive values of n. The author constructs a simple determinant whose value is V_n and also evaluates various other determinants of special type. *A. W. Goodman* (Lexington, Ky.).

Borofsky, Samuel. Solvability of quartics by means of square roots. Amer. Math. Monthly **57**, 248–250 (1950).

The author derives a well-known criterion for the roots of a quartic equation to be constructible with ruler and compasses [see, e.g., Vahlen, Acta Math. **21**, 287–299 (1897), p. 292]. *F. A. Behrend* (Melbourne).

Perron, Oskar. Ein Beweis des Fundamentalsatzes der Algebra im Reellen. Ann. Mat. Pura Appl. (4) **28**, 183–187 (1949).

This proof is obtained by modifying Cauchy's proof by working with congruences mod (y^2+1). The only tools required are Taylor's theorem for polynomials in one variable and the existence of a minimum of a polynomial in two variables [see also H. Schmidt, Jber. Deutsch. Math. Verein **46**, 64–67 (1936)]. *O. Todd-Taussky.*

Niven, Ivan. Extension of a topological proof of the fundamental theorem of algebra. Amer. Math. Monthly **57**, 246–248 (1950).

The paper attempts to give an alternative proof of the "fundamental theorem of algebra" for quaternions, as established by S. Eilenberg and I. Niven [Bull. Amer. Math. Soc. **50**, 246–248 (1944); these Rev. **5**, 169]. The present proof closely follows a similar proof of the ordinary fundamental theorem of algebra published by B. H. Arnold [Amer. Math. Monthly **56**, 465–466 (1949); these Rev. **11**, 195]. Both proofs attempt to deduce these theorems from the Brouwer fixed point theorem for the 4-cell and the 2-cell respectively. Unfortunately, both proofs contain an error; the functions $g(z)$ considered by Arnold and Niven are not continuous for $z=0$. *S. Eilenberg* (New York, N. Y.).

van der Corput, J. G. Sur les fonctions symétriques. Nederl. Akad. Wetensch., Proc. **53**, 703–717 = Indagationes Math. **12**, 216–230 (1950).

This lecture is a report on the author's new method for expressing a simple sum $\sum z_1^{h_1}\cdots z_r^{h_r}$ in terms of the elementary symmetric functions [see van der Corput, Math. Centrum Amsterdam, Scriptum no. **3** (1949); these Rev. **11**, 74]. The method uses certain differential operators with properties which make it possible to find the coefficients easily in numerical examples and which reveal symmetries between them. A new proof for the fundamental theorem of symmetric functions is also given. *O. Todd-Taussky.*

Brauer, Alfred. A criterion for a common root of k algebraic equations. Amer. Math. Monthly **57**, 322–324 (1950).

Let $f, f_1, \cdots, f_n$ be polynomials with complex coefficients and a single indeterminate. This paper specifies two cases in which the vanishing of the resultant of f and an appropriately defined polynomial F constitutes a necessary and sufficient condition that $f, f_1, \cdots, f_n$ have a common root. In the first case, the roots of f are assumed to lie on an interval of the real axis and $f_1, \cdots, f_n$ to be nonnegative on that interval and F is taken as $f_1+\cdots+f_n$. In the second case, the roots of f are assumed real and $F=f_1f_1^*+\cdots+f_nf_n^*$, where f_i^* is the polynomial whose coefficients are the complex conjugates of the coefficients of f_i. A general criterion of the same nature, valid simply under the assumption (implicit in the author's work) that the initial of f is not zero, is known and may be obtained by taking F equal to $\lambda_1f_1+\cdots+\lambda_nf_n$, where the λ's are arbitrary parameters [van der Waerden, Moderne Algebra, vol. 2, 1st ed., Springer, Berlin, 1931, pp. 6–7]. The criteria of the paper, when applicable, may involve simpler calculations in numerical cases. *J. M. Thomas* (Durham, N. C.).

Császár, Ákos. Sur les formes quadratiques positives. Publ. Math. Debrecen **1**, 186–188 (1950).

This paper gives a simple inductive proof, using only fundamental properties of determinants, of the well-known criterion that a quadratic form be positive. *B. W. Jones* (Boulder, Colo.).

Szele, T. Über die Klassifikation der quadratischen Formen. Publ. Math. Debrecen **1**, 189–192 (1950).

This paper gives a simple inductive proof, using linear forms, of the well-known criterion that a quadratic form be positive and a similar criterion that a form be positive semi-definite. *B. W. Jones* (Boulder, Colo.).

Abstract Algebra

Riguet, Jacques. Quelques propriétés des relations difonctionnelles. C. R. Acad. Sci. Paris **230**, 1999–2000 (1950).

In the notation of the author's previous work [Bull. Soc. Math. France **76**, 114–155 (1948); these Rev. **10**, 502], the difunctional closure of a binary relation R is the relation $R\bar{R}^{-1}R$, where $\bar{R}$ denotes the union of the relations R^i, $i=1, 2, \cdots$. A relation is difunctional if and only if it is closed under difunctional closure. Some related results are stated. *P. M. Whitman* (Silver Spring, Md.).

Kurepa, Djuro. The concept of a binary relation. Equivalence relation. Order relations. Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 53–58 (1949). (Croatian)

The equivalence relation between elements of a set S is discussed as a two-valued function in the product set $S\times S$. *W. Feller* (Princeton, N. J.).

Croisot, Robert. Axiomatique des treillis semi-modulaires. C. R. Acad. Sci. Paris **231**, 12–14 (1950).

The author considers thirteen different conditions for semi-modularity which are equivalent in lattices of finite length, but not in general lattices. He determines almost all implication relations connecting these conditions in (i) the class of all lattices, (ii) the class of all lattices satisfying the ascending chain condition, (iii) the class of all lattices satisfying the descending chain condition. In particular, he largely solves problem 44 of the reviewer's "Lattice Theory" [Amer. Math. Soc. Colloquium Publ., vol. 25, revised ed., New York, 1948; these Rev. **10**, 673]. *G. Birkhoff.*

Croisot, Robert. Axiomatique des treillis modulaires. C. R. Acad. Sci. Paris **231**, 95–97 (1950).

Various pairs (especially dual pairs) of the thirteen conditions and their duals in the paper reviewed above are shown to be (necessary and) sufficient for a lattice to be modular. The cases when a chain condition does or does not hold are distinguished. *G. Birkhoff* (Cambridge, Mass.).

Schwan, W. Zusammensetzung von Schwesterperspektivitäten in Verbänden. Math. Z. **52**, 150–167 (1949).
Schwan, W. Ein Homomorphiesatz der Theorie der Verbände. Math. Z. **52**, 193–201 (1949).

The author continues to develop his partial generalization of the concept of perspectivity from modular lattices to general lattices [Math. Z. **51**, 126–134, 346–354 (1948); these Rev. **10**, 4, 348]. He now treats "sister perspectivities," defined by the correspondences (i) $x \to b \cap x$, $y \to a \cup y$, and (ii) $y \to b \cup y$, $x \to a \cap x$. Here (i) defines an order-isomorphism between the set $a \vee b/a$ of all $a \cup (b \cap x)$ and the set $a/a \wedge b$ of all $b \cap (a \cup y)$, etc. In particular, writing c for $a \cup b$ and d for $a \wedge b$, if the four sets c/a, c/b, a/d, b/d are sublattices, then so is the set $(c/a \wedge c/b)/(a/d \vee b/d)$.

G. Birkhoff (Cambridge, Mass.).

Mousinho, Maria Laura. Modular and projective lattice. Summa Brasil. Math. **2**, no. 7, 95–112 (1950).

The author analyses the relationships of modular lattices and projective spaces, along the same general lines as Frink [Trans. Amer. Math. Soc. **60**, 452–467 (1946); these Rev. **8**, 309] except that complementation is not assumed. Let L be a modular lattice with 0; take as the set of points of a space a set E of atoms of L; take as lines the sets of atoms in E preceding the join of two distinct points; then the space is projective, and the lattice of all its subspaces is called the projective lattice determined by E. A set of maximal dual ideals of a modular lattice is a family of points of a projective space. A lattice is said to be disjunctive if $A \not\subset B$ implies that for some X, $0 \neq X \subset A$ and $B \cap X = 0$. A modular disjunctive lattice L is isomorphic to a sublattice of a projective lattice. A necessary and sufficient condition for a lattice to be isomorphic to the family of all subspaces of a projective space is that it be complete, atomic, disjunctive, modular, and satisfy the condition that whenever a point is contained in a join of points then it is contained in the join of a finite number of them. *P. M. Whitman.*

Jordan, Pascual. Zur Quanten-Logik. Arch. Math. **2**, 166–171 (1950).

Let M be any orthocomplemented modular lattice. The author shows that, if a, a', b generate a distributive sublattice, then so do a, a', b, b'. Pairs a, b with this property may be called permutable since, in the quantum-theoretic interpretation, a and b correspond to permutable Hermitian projections. The author shows that the elements permutable with any a form an (orthocomplemented) sublattice.

G. Birkhoff (Cambridge, Mass.).

Jordan, Pascual. Zur Axiomatik der Quanten-Algebra. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1950**, no. 5, 125–131 (1950).

The author discusses various postulate systems for quantum algebra, based on the idea that addition and squaring should be taken as the fundamental operations. "Multiplication" can be defined from these operations in several ways, one of which leads to a "Fastkörper" with one-sided distributivity. The determinations of "Fastkörper" with real centers by Kalscheuer [Abh. Math. Sem. Hansischen Univ. **13**, 413–435 (1940); these Rev. **1**, 328] give instructive examples which illustrate the relation between the various postulate systems. *G. Birkhoff* (Cambridge, Mass.).

Jordan, Pascual. Zur Theorie der Cayley-Grössen. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1950**, no. 1, 1–7 (1950).

The real Cayley-Dickson division algebra D has an involution $x \to \bar{x}$ such that $N(x) = \bar{x}x$ and $T(x) = x + \bar{x}$ are real numbers called the norm and trace of x respectively. If $X = (x_1, \cdots, x_n)$ is a vector with coordinates x_i in D we may define $N(X) = N(x_1) + \cdots + N(x_n)$. If also $Y = (y_1, \cdots, y_n)$ we define an inner product by $Q(X, Y) = \frac{1}{2}\sum_{\nu,\mu=1}^{n} T(\bar{x}_\nu x_\mu \cdot \bar{y}_\nu y_\mu)$. The author proves the relation $0 \leqq Q(x, y) \leqq N(X)N(Y)$ in the case $n = 2$, a result having consequences in the Moufang geometry. The author also proves the inequality for $n = 3$ under the assumption that each vector has coordinates in an associative subalgebra of D. *A. A. Albert* (Chicago, Ill.).

Albert, A. A. A note on the exceptional Jordan algebra. Proc. Nat. Acad. Sci. U. S. A. **36**, 372–374 (1950).

If $\mathfrak{A}$ is an associative algebra over a field $\mathfrak{F}$, then $\mathfrak{A}$ may be considered as a vector space over $\mathfrak{F}$ in which there is defined an associative multiplication xy. If $\mathfrak{F}$ has characteristic other than two, the same vector space may be used to define an algebra $\mathfrak{A}^{(+)}$ relative to the new operation $\frac{1}{2}(xy + yx)$. This algebra is a Jordan algebra. The author has previously shown [Ann. of Math. (2) **35**, 65–73 (1934)] that the Jordan algebra of all three-rowed Hermitian matrices with elements in the simple eight-dimensional Cayley algebra is not isomorphic to a subalgebra of any $\mathfrak{A}^{(+)}$ of finite dimension. In the present note, a new proof of this result is obtained but without any restriction on the dimension of $\mathfrak{A}$. *N. H. McCoy* (Northampton, Mass.).

Robinson, A. On non-associative systems. Proc. Edinburgh Math. Soc. (2) **8**, 111–118 (1949).

The paper is concerned with an abstract study of the arithmetic of the shapes of nonassociative products. Consider a mathematical system $(F, +)$ consisting of a set F and an operation $+$ on FF to F. If c and a are in F then a is said to be a component of c if $c = a + b$ or $c = b + a$ for b in F. Call c irreducible in F if c has no component in F. Then $(F, +)$ is called a forest if $a + b = c + d$ implies that $a = c$ and $b = d$, and every nonempty subset F' of F contains at least one element irreducible in F'. The order of a forest is the (cardinal) number π of irreducible elements in it, and the author shows that two forests of equal order are isomorphic. If $\pi = 1$ then $(F, +)$ is called simple and its only automorphism is the identity automorphism. Additional results on commutative and n-ary forests are obtained.

A. A. Albert (Chicago, Ill.).

Lyapin, E. S. Simple commutative associative systems. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 275–282 (1950). (Russian)

In this paper, "system" denotes a mathematical system with one operation satisfying both the associative and commutative laws. For other definitions, see an earlier paper of the author [same vol., 179–192 (1950); these Rev. **11**, 575]. One portion of the paper classifies all systems containing no nontrivial (1) ideals, or (2) normal complexes, or (3) normal subsystems. These three classifications combined with the above mentioned paper lead to the following con-

cluding remarks. A homomorphism is proper provided it is neither annihilating (that is, the image set contains but one element) nor an isomorphism. A system possessing no proper homomorphism is called simple. Cyclic groups of prime order and systems of order one or two are the only simple systems. A system $\mathfrak{A}$, not a group, possesses a non-annihilating homomorphism φ such that $\varphi\mathfrak{A}$ contains a null; a system $\mathfrak{A}$, which contains a null and has order greater than two, possesses a proper homomorphism φ such that $\varphi\mathfrak{A}$ contains a null. For a system $\mathfrak{A}$ which contains a null and every element of which is nilpotent, and only for such systems, the sole homomorphism φ such that $\varphi\mathfrak{A}$ contains an identity element is the annihilating homomorphism.

R. A. Good (College Park, Md.).

Fuchs, L. A note on the idealizer of a subring. Publ. Math. Debrecen **1**, 160–161 (1950).

L'auteur appelle idéalisateur d'un sous-anneau T d'un anneau commutatif R le plus grand sous-anneau S de R dans lequel T est un idéal, c'est-à-dire, l'ensemble $T:T$ des $x \varepsilon R$ tels que $xT \subset T$. Il montre que pour qu'un anneau R soit intégralement clos, il faut et il suffit que $A:A=R$ pour tout idéal A de type fini de R, contenant au moins un élément non diviseur de 0. Il prouve aussi que si P est un idéal maximal dans un anneau d'intégrité noethérien R, tel que $P:P=R$, on a $P \cdot P^{-1}=R$. *J. Dieudonné* (Nancy).

Rédei, L., und Szele, T. Algebraisch-zahlentheoretische Betrachtungen über Ringe. II. Acta Math. **82**, 209–241 (1950).

This is a continuation of an earlier investigation [Acta Math. **79**, 291–320 (1947); these Rev. **9**, 407]. The authors realize that the methods successfully employed there in the case of the ring of residues mod p^s will fail in general. Moreover, they realize that the method is not to be blamed, but that the formulation of the problem of "polynomial representation of R-functions" has to be clarified. If R is a commutative ring an R-function $\bar{f}(x)$ is one for which $\bar{f}(x) \varepsilon R$ for $x \varepsilon R$. The coefficients of the "representing" polynomial $f(x)$ are to be taken from a ring T, and the authors convince themselves that T is best assumed commutative. The main point is that x is to range in general over a subring S of T only. The ring S contains also the range of $f(x)$ and R is homomorphic with S. If $x \varepsilon S$ corresponds to $\bar{x} \varepsilon R$ under this homomorphism, then $f(x)$ represents $\bar{f}(\bar{x})=\overline{f(x)}$. The ring T is called a primitive ring for R, and R its derived ring; the ring S is called an intermediary ring. If all R-functions are represented by T-polynomials, then T is called a representation ring for R. Such a ring rarely exists. The best possible solution of the problem seems achieved by finding an optimum primitive ring which furnishes a polynomial representation for all R-functions which have polynomial representations in some primitive ring. With the help of these definitions a number of theorems are proved, the most important one being that only rings of characteristic p^s can have a representation ring. *O. Todd-Taussky.*

Levitzki, J. A theorem on polynomial identities. Proc. Amer. Math. Soc. **1**, 334–341 (1950).

The main theorem asserts that if S is a ring satisfying a polynomial identity of degree d, then the $[d/2]$th power of any nilpotent element generates a nilpotent ideal in S. From this it is further deduced that if S is a nil ring, it coincides with its lower radical in the sense of Baer [Amer. J. Math. **65**, 537–568 (1943); these Rev. **5**, 88], and that the radical series from 0 to S has at most $\log d/\log 2$ steps. [In the meantime, Amitsur, in an unpublished paper, has shown that it has at most 2 steps.] *I. Kaplansky.*

***Bourbaki, N. Éléments de mathématique. XI. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre IV: Polynomes et fractions rationnelles. Chapitre V: Corps commutatifs.** Actualités Sci. Ind., no. 1102. Hermann et Cie., Paris, 1950. ii+219+iii pp. 1800 francs.

The present fascicule is the fourth in the algebra book of the Bourbaki series; the first three are chapter I Structures algébrique, chapter II Algèbre linéaire, and chapter III Algèbre multilinéaire [Actualités Sci. Ind., nos. 934, 1032, 1034, Hermann, Paris, 1942, 1947, 1948; these Rev. **6**, 113; **9**, 406; **10**, 231]. Chapters IV and V present a carefully worked out development of the theory of polynomials and (commutative) fields, including the topics usually included in modern algebra books under this heading, and several others. Since the present work is part of a unified treatise, it is possible in many instances to give short proofs and enlightening comments by referring to other parts of the series, at the same time making it necessary for the reader to know at least some of the definitions and results of other parts. For example, the proposition that for any field K and any family (E_α) of field extensions of K there exists a field extension E of K and a family (u_α) where, for each α, u_α is an isomorphism over K of E_α into E and $E=K(\bigcup_\alpha E_\alpha)$, is proved by taking in the tensor product $F=\otimes_\alpha E_\alpha$ a maximal ideal $\mathfrak{A}$ and forming the algebra $F/\mathfrak{A}$; this algebra is obviously a field extension of K (the required extension E), and it is easy to see that the canonical homomorphism of E_α into $F/\mathfrak{A}$ is an isomorphism (the required isomorphism). The existence and essential uniqueness of an algebraic closure of K are then derived as simple corollaries of this proposition. Additional features which are noteworthy because of their novelty in books on algebra include: a rather full discussion of formal power series; the use of the concepts of linear disjointness and algebraic disjointness of two field extensions of a field (concepts introduced in A. Weil's Foundations of Algebraic Geometry [Amer. Math. Soc. Colloquium Publ., v. 29, American Mathematical Society, New York, N. Y., 1946; these Rev. **9**, 303]); distinction between normal extension of a field K (each irreducible polynomial in $K(X)$ which has one root in the extension E decomposes into linear factors in $E[X]$), and Galois extension of K (every element of the extension E which is invariant under every automorphism of E over K is in K); using a definition of separable that applies to transcendental as well as algebraic extensions (and, among other things, using this definition to obtain a criterion for the existence of a separating transcendence base); a discussion of derivations of a field. Galois theory is developed more or less à la Artin (but with variations), and there is an appendix treating Krull's Galois theory of extensions of infinite degree. The text is accompanied by a set of exercises of varying degrees of difficulty, with and without hints; some of these illuminate the theorems of the text by providing proofs or counterexamples of generalizations; others contain theorems not included in the text, both well known (e.g., Lüroth's theorem, and the commutativity of every finite field) and otherwise. *E. R. Kolchin* (New York, N. Y.).

Kaplansky, Irving. The Weierstrass theorem in fields with valuations. Proc. Amer. Math. Soc. **1**, 356–357 (1950).

The author uses the methods of a previous investigation of topological rings with ideal neighborhoods of 0 [Amer. J.

Math. 69, 153–183 (1947); these Rev. 8, 434] to generalize a theorem of Dieudonné concerning the field of p-adic numbers [Bull. Sci. Math. (2) 68, 79–95 (1944); these Rev. 7, 111] to division rings F with a valuation whose value group is a subgroup of the real numbers. Let C be the ring of continuous functions from a totally disconnected locally compact space to F vanishing at ∞. If C is topologized by uniform convergence, then it coincides with any closed subring admitting left multiplication by constant functions and containing a function which vanishes at a given x, and not at a given y. *O. Todd-Taussky* (Washington, D. C.).

Dynkin, E. B. The structure of semi-simple algebras. Amer. Math. Soc. Translation no. 17, 143 pp. (1950).

Translated from Uspehi Matem. Nauk (N.S.) 2, no. 4(20), 59–127 (1947); these Rev. 10, 350.

Volovel'skaya, S. N. Analytic functions in non-semisimple associative linear algebras. Zapiski Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) 19, 153–159 (1948). (Russian)

The author asserts that previous accounts of function theory in algebras are inadequate in case there is a radical, and presents a theory modelled on her previous account of algebras of order 3 [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 143–157 (1940); these Rev. 3, 85]. She confines herself to algebras of the "first category" in the sense of Cartan, and uses a "normalized basis." (In modern terminology, an algebra is of the first category if it is commutative modulo its radical).

I. Kaplansky (Chicago, Ill.).

***Ritt, Joseph Fels. Differential Algebra.** American Mathematical Society Colloquium Publications, Vol. XXXIII, American Mathematical Society, New York, N. Y., 1950. viii+184 pp. $4.40.

This book is an exposition of the work by the author and his collaborators, H. W. Raudenbush, W. C. Strodt, E. R. Kolchin, H. Levi, E. Gourin, R. M. Cohn, and others, on differential polynomials, differential ideals, and differential manifolds, or, in short, on differential algebra. It is in a sense a revision of the author's book, Differential Equations From the Algebraic Standpoint [Amer. Math. Soc. Colloquium Publ., v. 14, New York, 1932], but it is far more than that.

Chapter I begins by defining a differential polynomial (d.p.), in the differential indeterminates $y_1, \cdots, y_n$ over an (abstract) differential field of characteristic 0, its class (largest j such that y_j appears), and its order with respect to a y_i (largest j such that the jth derivative y_{ij} of y_i appears in it). The d.p. A_2 is of higher rank in y_i than the d.p. A_1 either when A_2 is of higher order than A_1 in y_i, or when they are of the same order, say j, in y_i and A_2 is of higher degree than A_1 in y_{ij}. The d.p. A_2 is of higher rank, or simply "higher," than A_1 either when A_2 is of higher class than A_1, or when they are of a same class i and A_2 is of higher rank than A_1 in y_i. If A_1 is of class $i>0$, A_2 is said to be reduced with respect to A_1 if A_2 is of lower rank than A_1 in y_i. A system $A_1, \cdots, A_r$ of d.p. is a chain if either (a) $r=1$, $A_1 \neq 0$, or (b) $r>1$, A_1 is of positive class and, for $j>i$, A_j is of higher class than A_i and reduced with respect to A_i. One chain is of higher rank than a second chain when it is so lexicographically, provided that a void term is regarded as higher than any d.p. This partially orders chains, and (the number n of indeterminates being finite) the minimum condition prevails. If Σ is a set of d.p., a chain of lowest rank made up of d.p. in Σ is called a characteristic set of Σ. If a d.p. A is of class $i>0$ and order j in y_i, we call $\partial A/\partial y_{ij}$ the separant of A. The coefficient of the highest power of y_{ij} in A is the initial of G. If we multiply a d.p. G by suitable powers of the separants and initials of the A_i and subtract a linear combination of the A_i and their derivatives, with d.p. for coefficients, then the remainder R will be reduced with respect to the chain $A_1, \cdots, A_r$, that is, with respect to each A_i; R is called the remainder of G with respect to the chain.

By an ideal (of d.p.) is meant one which contains the derivative of each d.p. in it. It is called perfect if, whenever a power of a d.p. A is contained in it, A is contained in it. The perfect ideal generated by a set Λ of d.p. is denoted by $\{\Lambda\}$. A finite subset Φ of Σ is a basis of Σ if $\{\Phi\} \supseteq \Sigma$. By virtue of the minimum condition and the reduction process, every system of d.p. has a basis. Every perfect ideal is represented as the intersection of a finite number of prime ideals, and, in fact, of essential prime divisors (not containing any other prime divisor). The decomposition of an ideal into relatively prime ideals, the behaviour of a prime ideal with respect to the adjunction of a new indeterminate, and extension of the ground field are also considered.

Chapter II treats (algebraic) differential manifolds. Every manifold is the union of a finite number of irreducible manifolds; essential irreducible components (or simply components) correspond to the essential prime divisors of the associated ideal. A generic zero of a prime ideal Σ is a zero of Σ such that every d.p. which is annulled by it is contained in Σ; one can be constructed just as in the case of an algebraic variety. The generalization of the Hilbert theorem on zeros states that if G is a d.p. holding (i.e., annulled by every zero of) a system $F_1, \cdots, F_p$, then some power of G is a linear combination of the F_q and their derivatives. Next let F be an algebraically irreducible (i.e., not factorable into two d.p. with positive classes) d.p. (of positive class). Denoting the separant of A by S, the totality Σ_1 of d.p. A such that $SA \epsilon \{F\}$ can be proved to be the unique prime divisor of $\{F\}$ which does not contain S; the manifold of Σ_1 is called the general solution of F.

Given a nontrivial prime ideal Σ, there exists a subsystem $u_1, \cdots, u_q$ of $y_1, \cdots, y_n$ such that no nonzero d.p. of Σ involves the u alone and such that, given any y_j not among the u, there is a nonzero d.p. of Σ in y_j and the u alone. Renumbering the indeterminates we can assume that $u_1, \cdots, u_q$, $y_1, \cdots, y_p$ is our set of indeterminates. Then $u_1, \cdots, u_q$ is called a parametric set of indeterminates for Σ. On introducing a new indeterminate w, there is obtained a certain prime ideal Ω such that the d.p. of Ω free of w are precisely those of Σ, and, moreover, if $A, A_1, \cdots, A_p$ is a characteristic set of Ω with respect to $u_1, \cdots, u_q, w, y_1, \cdots, y_p$, with A algebraically irreducible, then the A_i express the y_i rationally in terms of $w, u_1, \cdots, u_q$ and their derivatives. Thus the determination of the manifold of Σ is made to depend on the determination of the general solution of $A=0$, this equation being called a resolvent of the prime ideal Σ. The number q is called the dimension of Σ, while the order of the resolvent is called the order of Σ with respect to u. For a single indeterminate y every irreducible manifold is the general solution of a d.p., and this fact is used also in obtaining a generalization of Lüroth's theorem which asserts that a subfield of a (differential) field generated by a single indeterminate is also generated by a single element. In this connection, the manifold of an algebraically irreducible d.p.

F may be analysed and it can be seen that its components are all of dimension $n-1$ and are general solutions of certain d.p. $A_1, \cdots, A_p$. For a proof one shows that F has a zero of the form $y_1=\eta_1+\zeta_1 c+\varphi_2 c^2+\cdots$, $y_i=\eta_i+\zeta_i c$ $(i=2, \cdots, n)$, where $(\eta_1, \cdots, \eta_n)$ is a generic zero of some component under consideration whose existence is proved by a certain polygon process. The problem of determining the A_i can be answered by the low power theorem: Let the d.p. F and A be of class n, of respective orders m and l in y_n, and with A algebraically irreducible. There exist a t and an r (>0) such that S^tF has a representation as a linear combination of r distinct monomials in A and its derivatives up to the $(m-l)$th, the coefficients being d.p. of orders not greater than l in y_n and not divisible by A. For the general solution of A to be a component of F, it is necessary and sufficient that such an expression contain a term free of proper derivatives of A and of lower degree with respect to A and its derivatives than every other term. Next, the exponent of an ideal Σ is the least number p such that $\{\Sigma\}^p \subseteq \Sigma$ (or ∞ if such does not exist). Without going into the details of Kolchin's investigation, the book presents an example for illustration.

In chapter IV the author treats systems of algebraic equations and their manifolds, emphasizing algorithmic methods. The results are applied in chapter V to obtain an algorithmic resolution of a system of d.p. into prime ideals, after reducing the problem to the case of algebraic equations. Further, algorithmic or constructive approaches to the resolvent, theorem of zeros, and ideal decomposition are given.

Chapter VI treats the case where one deals with analytic functions; to be exact, one deals with a ground field consisting of functions meromorphic in a certain open region, and understands by an analytic zero a set of functions analytic in an open region contained in the given region and satisfying the differential equations. This case has been studied at times in the earlier chapters, where it has been shown that it may be handled in exactly the same manner as in the abstract case if we deal with analytic zeros and restricted manifolds (consisting of analytic zeros). The chapter starts with the approximation theorem: If Σ is a (nontrivial) prime ideal and a d.p. B is not in Σ then every zero of Σ adheres to (can be approximated to any degree by) those which do not annul B. In particular, the manifold of Σ consists of normal (i.e., not annulling separants) zeros of a characteristic set and of zeros adhering to them. An analytic proof is given of the low power theorem. Sequence, sum, product, and derivative of manifold are considered briefly. Chapter VII is concerned with intersection of manifolds. It begins by showing that no regularity prevails for the dimension of the intersection as in the case of algebraic manifolds. As for the orders of components of the intersection, with respect to a given parametric set, there is a certain bound; the result can be regarded as a counterpart of Bezout's theorem in algebraic geometry. It is also proved that a zero of a d.p. F contained in more than one component of F annuls $\partial F/\partial y_{ij}$. Further, an analogue of a theorem of Kronecker is proved: If the ground field contains a non-constant element, every system is equivalent to a system of $n+1$ d.p.

In chapter VIII partial d.p. are considered and Riquier's existence theorem is proved. It begins by introducing marks for partial derivatives of y's and thus effecting a (complete) ordering of them. Consider a system of differential equations which express some derivatives as analytic functions of others, and call it orthonomic if for each equation the derivatives involved on the right side are lower in the ordering than the left side, if the left sides of any two equations are distinct, and if a derivative of a left side never appears in the right side of any equation. It is called passive when it satisfies the integrability conditions. Now the theorem asserts that a passive orthonomic system has a unique solution for any given initial determinations. Chapter IX develops the theory of ideals and manifolds of partial d.p. By virtue of our ordering according to marks, we can introduce rank, chain, characteristic set, separant, initial, etc., more or less as before. Similarly for the theories of reduction, general solutions, and the low power theorem, although of course some modifications are needed. On using Riquier's existence theorem, a characterization of characteristic sets of prime ideals, an elimination theory for systems of (algebraic) partial differential equations, as well as the theorem of zeros, are given for the analytic case.

The book closes with an appendix, in which the author raises some problems. For instance, construction of a theory of d.p. possessing the scope of the Lasker-Noether theory of polynomial ideals by introducing some combinations other than intersection and product, study of d.p. over a field of characteristic $p \neq 0$, study of differential power series, study of singular solutions of partial differential equations, etc., are proposed.

Needless to say the book is much larger than the author's older monograph and includes many advances made since then by the author and his followers. For instance, the algebraic theory of perfect ideals, with the basis theorem as its main tool, constructed by Raudenbush at the author's suggestion, and Levi's algebraic approach to the author's low power theorem are both new. Also, the older book treated the analytic case exclusively. Cohn's constructive proof of the theorem of zeros, Gourin's result on embedded manifolds, and Strodt's work on sequences of manifolds are other new features. We have alluded to Kolchin's work on exponents above, but miss his recent work on the Picard-Vessiot theory. The exposition is both thorough and clear.

T. Nakayama (Urbana, Ill.).

Mikusiński, Jan G.-. Sur l'unicité des solutions de quelques équations différentielles dans les espaces abstraits. Ann. Soc. Polon. Math. 22, 157–160 (1949).

Let A be a ring without divisors of zero. The author proves uniqueness theorems for the differential equations $ax'(\lambda)+bx(\lambda)=c(\lambda)$, $ax''(\lambda)+bx(\lambda)=c(\lambda)$, each with appropriate one-point initial conditions. Here $x(\lambda)$ and $c(\lambda)$ are functions from the open real interval (α, β) to A, and $a, b \in A$, $a \neq 0$. Differentiation may be any formally defined process obeying the usual rules for sums, differences and products, and having the two further properties: (1) if μ is constant and $x(\lambda)$ is differentiable on (α, β) then $y(\lambda)=x(\mu-\lambda)$ is differentiable, with $y'(\lambda)=-x'(\mu-\lambda)$ when $\alpha<\mu-\lambda<\beta$; (2) $x'(\lambda)=0$ in (α, β) is equivalent to the constancy of $x(\lambda)$ in (α, β). For the first order case it is sufficient to show that $ax'(\lambda)+bx(\lambda)=0$, $x'(\lambda_0)=0$ has the unique solution $x(\lambda)=0$. The author shows that if $ax'(\lambda)+bx(\lambda)=0$ and $x(\mu)=0$ for a certain μ of (α, β), then $x(\lambda)=0$ when $\frac{1}{2}(\alpha+\mu)<\lambda<\frac{1}{2}(\beta+\mu)$. The uniqueness theorem then follows by repeated application of the reasoning. For fixed λ in (α, β) define $y(\mu)=ax(\mu)x(2\lambda-\mu)$. One readily shows $y'(\mu)=0$, and hence $x(\mu)x(2\lambda-\mu)=x(\lambda)x(\lambda)$ if $\alpha<2\lambda-\mu<\beta$. Then $x(\mu)=0$ implies $x(\lambda)=0$ if $\frac{1}{2}(\alpha+\mu)<\lambda<\frac{1}{2}(\beta+\mu)$. This argument can be adapted to the second order equation. The uniqueness theorem may fail if A has divisors of zero.

A. E. Taylor (Los Angeles, Calif.).

Drobot, S., et Mikusiński, J. G.-. Sur l'unicité des solutions des quelques équations différentielles dans les espaces abstraits. II. Studia Math. 11, 38–40 (1949).

The authors extend the uniqueness theorem of Mikusiński for first order equations [cf. the preceding review] to the general linear differential equation of nth order with single point boundary conditions. The proof is by induction.

A. E. Taylor (Los Angeles, Calif.).

Theory of Groups

Piccard, Sophie. Sur les groupes imprimitifs. C. R. Acad. Sci. Paris 231, 14–16 (1950).

Consider an imprimitive permutation group G_1 on p_1 symbols which can be subdivided into p_2 sets of imprimitivity E_i^1, $i=1, 2, \cdots, p_2$. If these sets E_i^1 are permuted by the substitutions of G_1 according to a permutation group G_2, p_1/p_2-isomorphic with G_1, then G_2 is called the first associated group of G_1. Treating G_2 in like manner one can construct a sequence of $m \geqq 2$ associated groups $G_1, G_2, \cdots, G_m$, where G_m is primitive of degree p_m. If S_1 is a substitution of G_1, and S_i the corresponding substitution of G_i, then the set S_i, $i=1, 2, \cdots, m$, is said to form a complete set associated with S_1. In terms of this set the author states several properties of G_1 without giving proofs. *G. de B. Robinson.*

Devidé, Vladimir. Einige Eigenschaften von Gruppen in welchen mehrere Operationen definiert sind. Hrvatske Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 4, 97–103 (1949). (German. Croatian summary)

Consider a group G with composition ab and unit e. Suppose that there is defined a second multiplication $a\sigma b$ of the elements in G subject to the rule: (*) $(a\sigma b)\sigma c = a\sigma(b\sigma c)$. Then $b\sigma c = b(e\sigma e)c$ for every b and c; and under σ-multiplication G is a group isomorphic to the original group G. Conversely, if k is an element in G, and if we define a multiplication σ by the rule $b\sigma c = bkc$, then σ meets the requirement (*). If σ' and σ'' are two multiplications meeting the requirement (*), then $(a\sigma' b)\sigma'' c = a\sigma'(b\sigma'' c)$.

R. Baer (Urbana, Ill.).

Szele, T. Über die Abelschen Gruppen mit nullteilerfreiem Endomorphismenring. Publ. Math. Debrecen 1, 89–91 (1949).

Assume that the Abelian group A contains elements of finite order. Then the endomorphism ring of A does not contain divisors of zero if, and only if, A is either cyclic of order p, a prime, or A is of type p^∞ [in the sense of Prüfer].

R. Baer (Urbana, Ill.).

Almeida Costa, A. On the endomorphisms of modules. Centro Estudos Mat. Fac. Ci. Pôrto Publ. no. 20 = Anais Fac. Ci. Pôrto 33,32 pp. (1948). (Portuguese)

This is an account, largely expository, of the ring R of endomorphisms of an Abelian group M with operators. The discussion is in the same spirit as chapter 4 of Jacobson's book [The Theory of Rings, American Mathematical Society, New York, 1943; these Rev. 5, 31]. The theorems are too numerous to describe in detail. The topics covered are: connection between ideals in R and subgroups of M, idempotents in R, chain conditions in M (descending, ascending, and both), structure of R in terms of that of M, case where M is homogeneous or completely reducible.

I. Kaplansky (Chicago, Ill.).

Baer, Reinhold. Free mobility and orthogonality. Trans. Amer. Math. Soc. 68, 439–460 (1950).

Let F be an ordered sfield, let P be the system of its positive numbers, and let V be an F-vector-space of dimension $n>2$. If $b_1, \cdots, b_k$ are independent vectors in V, the manifolds $Fb_1+\cdots+Fb_{k-1}+Pb_k$, $k=1, \cdots, h$ are said to form an h-dimensional chain of incident sub-half spaces of V. A group Γ of linear transformations on V is said to have the property (P.h) of h-dimensional free mobility if Γ is simply transitive on the set of all the h-dimensional chains of incident sub-half spaces of V. Such groups exist only for $h=1$ or $n-1$ or n. The group Γ has property (P.n) if and only if F is a Pythagorean field and there exists a symmetrical, positive definite form $f(x, y)$ from V to F such that Γ consists of all linear transformations on V which preserve the form f. The group ϕ has the property (P.$n-1$) if and only if ϕ is the subgroup of index 2 of a group with property (P.n). The groups with properties (P.n) and (P.$n-1$) are respectively the full and proper orthogonal groups. For $n=2$ there are additional groups with property (P.1), the result for (P.n) remains valid for (P.2) if certain additional assumptions are made on F. *R. M. Thrall.*

Racah, Giulio. Sulla caratterizzazione delle rappresentazioni irriducibili dei gruppi semisemplici di Lie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 108–112 (1950).

Let L be a semi-simple Lie algebra of rank l. The author shows the existence of certain independent operators, l in number, which permute with all the operators adj X, $X \in L$. With this, it is asserted, a simple proof by Casimir and van der Waerden [Math. Ann. 111, 1–12 (1935)] of the complete reducibility of the representations of the rotation group in 3-space becomes valid for arbitrary semi-simple groups. *P. A. Smith* (New York, N. Y.).

***Ma, S. T. Three-dimensional rotations.** National Peking University Semi-Centennial Volume. Mathematical, Physical and Biological Series, pp. 60–80, 1948.

The author reviews some aspects of the theory of representations of the real proper three-dimensional orthogonal group. He includes a brief review of the two component spinor representation of this group. *A. H. Taub.*

Gel'fand, I. M., and Naĭmark, M. A. The relation between the unitary representations of the complex unimodular group and its unitary subgroup. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 239–260 (1950). (Russian)

Detailed proofs are given of theorems previously announced [Doklady Akad. Nauk SSSR (N.S.) 63, 225–228 (1948); these Rev. 10, 282]. In this paper the authors confine themselves to the complex unimodular group.

I. Kaplansky (Chicago, Ill.).

Gel'fand, I. M., and Cetlin, M. L. Finite-dimensional representations of the group of unimodular matrices. Doklady Akad. Nauk SSSR (N.S.) 71, 825–828 (1950). (Russian)

Explicit formulas are given for the irreducible representations of the Lie algebra of all matrices of a given finite order, thereby determining the irreducible finite-dimensional representations of the full linear groups, from which the corresponding representations of the unimodular groups are trivially obtainable. Less explicit determination of these representations is classical. *I. E. Segal* (Chicago, Ill.).

Makar, Ragy H. On the analysis of the Kronecker product of irreducible representations of the symmetric group. Proc. Edinburgh Math. Soc. (2) **8**, 133–137 (1949).

Let $D(\lambda)$ denote the irreducible representation of the symmetric group of degree p defined by the partition (λ): $p=\lambda_1+\cdots+\lambda_p$, $\lambda_1>\cdots>\lambda_p\geqq 0$. Formulas are given for the analysis of the Kronecker product representation $D(\lambda)\times D(\mu)$ for all partitions (μ) of n such that $n-\mu_1\leqq 3$ and for all partitions (λ) of p. These formulas are expressed in terms of the operations of removing hooks of lengths 1, 2, and 3 from (λ) and of multiplication of Schur functions where one factor has weight not greater than 3. *R. M. Thrall.*

Vilenkin, N. Ya. Direct decompositions of topological groups. I, II. Amer. Math. Soc. Translation no. **23**, 109 pp. (1950).

Translation of papers which appeared in Rec. Math. [Mat. Sbornik] N.S. **19**(61), 85–154, 311–340 (1946); these Rev. **8**, 132, 312.

Fuchs, Ladislas. On partially ordered groups. Nederl. Akad. Wetensch., Proc. **53**, 828–834 = Indagationes Math. **12**, 272–278 (1950).

A systematic study is made of order-homomorphisms $\theta: G\to G_1$ between partially ordered groups which satisfy the Moore-Smith condition, or "directed groups." These θ correspond one-to-one to convex normal subgroups K, and to strengthenings of the natural order in G/K. It is noted that the union of two convex normal subgroups need not be convex. An "ideal" (generalizing the notion of l-ideal in a lattice-ordered group) is defined as such a K having the Moore-Smith property, and two theorems about such ideals are proved. *G. Birkhoff* (Cambridge, Mass.).

Hille, Einar. Lie theory of semi-groups of linear transformations. Bull. Amer. Math. Soc. **56**, 89–114 (1950).

The theory of Lie groups is extended in two directions. First, the existence of inverses is not required, so that one deals with a manifold (possibly with boundary) on which an associative binary operation (multiplication) is defined, with various smoothness properties, a Lie semi-group, say. Second, representations of Lie semi-groups by bounded operators on Banach spaces are investigated; the classical Lie theory can be considered to correspond to the case of the regular representation of the group. In the special case of commutative multiplication the situation has been the subject of an intensive study by the author [Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloquium Publ., vol. 31, New York, 1948; these Rev. **9**, 594] and others. In the present paper it is shown under various restrictions, perhaps the most serious of which is that the manifold of the Lie semi-group be a closed "octant" E_n^+ in Euclidean n-space, with the origin as an identity in the semi-group, that important parts of the Lie group and Abelian semi-group theories remain valid, when suitably reformulated, in the Lie semi-group case also.

Specifically, (1) in each direction into E_n^+ from the origin there passes a unique 1-parameter sub-semi-group, and every continuous such sub-semi-group arises in this fashion (and so is differentiable). (2) If T is a strongly continuous representation of the Lie semi-group G by operators, on a Banach space, of norms bounded by one, and if $x(\alpha)$, $\alpha\geqq 0$, is the 1-parameter sub-semi-group with direction $a\in E_n^+$, then there exists a closed operator $A(a)$ with dense domain $D(a)$ such that for $y\in D(a)$, $A(a)y=\lim_{\alpha\to 0}\alpha^{-1}(T(x(\alpha))-I)y$; and the $D(a)$ and the domains of the $A(a)A(b)$ have a dense common part, as a and b range over E_n^+. (3) In fact, $D(a)$ includes all y of the form $y=\int_{E_n^+}K(p)T(p)z\,dp$, where K is sufficiently smooth, z is arbitrary in the Banach space, and dp is the element of n-dimensional Lebesgue measure; and the restriction of $A(a)$ to the domain H of all such y has $A(a)$ as its closure. (4) For any a and b in E_n^+, $D(a+b)\supset D(a)\cap D(b)$ and for $y\in D(a+b)$,

$$A(a+b)y=A(a)y+A(b)y.$$

(5) Constants of structure γ_{ijk} can be defined by one of the usual methods, and satisfy the usual anti-symmetry and Jacobi condition. If e_i, $i=1, 2, \cdots, n$, are the usual basis elements in Euclidean space, then the following equation is valid: $(A_iA_j-A_jA_i)y=\sum\gamma_{ijk}A(e_k)y$ for $y\in H$. (The map $a\to A(a)$ is thus a kind of semi-Lie-algebra-homomorphism.)

In proving (1) a kind of Lipschitz condition similar to, but slightly stronger than, that used in the theory of Lie groups is assumed. For the other results, slightly stronger conditions are needed on the smoothness of the semi-group composition together with the assumption on the representation T that it be faithful in some neighborhood of the origin. A typical complication in this work is that although the bracket of two infinitesimal generators $A(a)$ and $A(b)$ is again a linear combination of infinitesimal generators, it need not itself be one (as the coefficients in its expression as a linear combination need not be positive).

I. E. Segal (Chicago, Ill.).

NUMBER THEORY

✱**Vinogradov, I. M. Osnovy teorii čisel. [Foundations of the Theory of Numbers].** 5th ed. Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1949. 180 pp.

The 4th edition [1944] was reviewed in these Rev. **7**, 413.

Leeuwenkuijl, M., and Schuh, Fred. Concerning odd magic squares. Nieuw Tijdschr. Wiskunde **37**, 289–294 (1950). (Dutch)

The authors have rediscovered a generalization of Bachet's rule [M. Kraïtchik, La Mathématique des Jeux, Stevens, Bruxelles, 1930, p. 128]. Their magic square of order $2n+1$ is constructed by placing the natural numbers successively, using a bishop's move of c steps in one diagonal direction from ξ to $\xi+1$ unless ξ is a multiple of $2n+1$, and a bishop's move of d steps in the other diagonal direction from 1 to $2n+2$. This is $\begin{pmatrix} c & -d \\ c & d \end{pmatrix}_{2n+1}$ in the notation of W. W. R. Ball [Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, p. 205; these Rev. **8**, 440]. Bachet's rule itself occurs when $c=d=1$.

H. S. M. Coxeter (Toronto, Ont.).

Obláth, Richard. Une remarque sur la progression arithmétique. Mat. Lapok **1**, 138–139 (1950). (Hungarian. Russian and French summaries)

The author proves that if $(a, d)=1$ then $a(a+d)(a+2d)$ is never a cube, fourth power, or fifth power.

P. Erdős (Aberdeen).

Cassels, J. W. S. The rational solutions of the diophantine equation $Y^2=X^3-D$. Acta Math. **82**, 243–273 (1950).

The equation of the title is a special case of

$$Y^2=X^3-CX-D,$$

where C and D are integers. The values of the parameter u corresponding to rational solutions form an additive group, $\mathfrak{U}$, when the usual parametrization $Y=\frac{1}{2}\wp'(u)$, $X=\wp(u)$ is employed. Weil [Acta Math. **52**, 281–315 (1929)] has given a far-reaching generalization of the theorem which states that this group has a finite basis. Using an algorithm which underlies Weil's theorem, the author gives a complete solution of $Y^2=X^3-D$ for all $|D|\leqq 50$ in the sense that he gives a complete basis for $\mathfrak{U}$. With one exception ($D=-15$) his results confirm those of Billing's thesis [Nova Acta Soc. Sci. Upsaliensis (4) **11**, 1–165 (1938)] in the range $|D|\leqq 25$. The proofs involve detailed developments of the arithmetical properties of the relevant cubic number fields. A table of class-numbers and units for all cubic fields $R(D^{\frac{1}{2}})$ with $|D|\leqq 50$ is included. Among the numerous applications the following may be cited. The equation $y^2=x^3-Dt^6$, where $t\neq 0$, $(x, t)=(y, t)=1$, $27\nmid D$ and D is sixth-power-free, has no integral solutions for $D=-11$, -39, $+43$, -46, -47 though rational solutions exist. Mordell [Proc. London Math. Soc. (2) **13**, 60–80 (1914)] has shown in the remaining cases $D=+21$, $+22$, $+29$, $+30$, $+38$, $+50$ (where the author obtains rational but no integral solutions) that no integral solutions exist. In general, however, the author's method appears unsuitable for discussing integral as opposed to rational solutions.

A. L. Whiteman (Los Angeles, Calif.).

Holzer, L. Minimal solutions of Diophantine equations. Canadian J. Math. **2**, 238–244 (1950).

The author proves the following theorem. If a, b, c are integers, $ab>1$, a, b, c relatively prime in pairs, and all free of squares, $-ab$ a quadratic residue of c, bc of a, ca of b, and if $F(x, y, z)=ax^2+by^2-cz^2$, we have nontrivial solutions of $F(x, y, z)=0$ with the inequalities $|x|<(bc)^{\frac{1}{2}}$, $|y|<(ca)^{\frac{1}{2}}$, $|z|<(ab)^{\frac{1}{2}}$. The proof is based on the following result by Hecke [Math. Z. **1**, 357–376 (1918), p. 375; **6**, 11–51 (1920), p. 38; see also Hasse, Jber. Deutsch. Math. Verein. **35**, 1–55 (1926), p. 32]. In a field let j be any integer ideal, α a number prime to j. Then there are infinitely many prime ideals (π) of the first degree with $\pi\equiv\alpha \pmod{j}$.

T. Nagell (Uppsala).

Rédei, L. Die Primfaktoren der Zahlenfolge 1, 3, 4, 7, 11, 18, ⋯. Portugaliae Math. **8**, 59–61 (1949).

The sequence $\mathfrak{U}$ of the title is understood to be defined by the relation $v_n=v_{n-1}+v_{n-2}$ $(n\geqq 3)$, with $v_1=1$, $v_2=3$. By a prime divisor of $\mathfrak{U}$ is meant a prime dividing v_n for some n. By simple considerations in the number field generated by $5^{\frac{1}{2}}$ the author proves: (1) if p is a prime congruent to 21 or 29 modulo 40, then a necessary and sufficient condition that p be a prime divisor of $\mathfrak{U}$ is that $(p|5)_4(5|p)_4=1$; (2) if p is a prime congruent to 1 or 9 modulo 40, then a sufficient condition that p be a prime divisor of $\mathfrak{U}$ is that $(p|5)_4(5|p)_4=-1$. Here $(b|q)_4$ is the so-called rational biquadratic residue symbol, which is defined for q an odd prime and b a quadratic residue of q and which has the value 1 or -1 according as the congruence $x^4\equiv a \pmod{p}$ is solvable or unsolvable.

P. T. Bateman (Urbana, Ill.).

De Bruijn, N. G. On the number of uncancelled elements in the sieve of Eratosthenes. Nederl. Akad. Wetensch., Proc. **53**, 803–812 = Indagationes Math. **12**, 247–256 (1950).

For $x>0$, $y\geqq 2$, let $\Phi(x, y)$ denote the number of positive integers less than or equal to x which have no prime factors less than y. The author derives estimates for $\Phi(x, y)$ in the regions $y^n\leqq x\leqq y^{n+1}$, $n=1, 2, \cdots$, which hold uniformly in n. His results and methods are improvements on those of Buchstab [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 1239–1246 (1937)]. Use is made of the Buchstab function $\omega(u)$ defined by $\omega(u)=1/u$, $1\leqq u\leqq 2$, and $(u\omega(u))'=\omega(u-1)$, $u\geqq 2$, where for $u=2$ the right-hand derivative has to be taken. Let $\pi(y)$ denote the number of primes less than or equal to y, $\mathrm{li}\, y$ the logarithmic integral, γ the Euler constant, C an absolute constant not necessarily the same each time it occurs. Put $\Phi(x, y)=x\prod_{p<y}(1-p^{-1})\cdot\psi(x, y)$. Then the main estimates are: (1) $|\psi(y^u, y)-e^\gamma \log y\int_1^u y^{t-u}\omega(t)dt|<CR(y)$, $u\geqq 1$, $y\geqq 2$, where $R(y)$ is a positive function satisfying $R(y)\downarrow 0$ for $y\to\infty$, $R(y)>1/y$ and $|\pi(y)-\mathrm{li}\, y|<yR(y)/\log y$, $\int_y^\infty|\pi(t)-\mathrm{li}\, t|t^{-2}dt<R(y)$, $y\geqq 2$;

(2) $|\psi(y^u, y)-1|<C\log^3 y\exp(-u\log u-u\log\log u+Cu)$, $1\leqq u\leqq 4y^{\frac{1}{2}}/\log y$, $y\geqq 2$;

(3) $|\psi(y^u, y)-1|<C\exp(-\frac{1}{3}u\log y)$, $u>4y^{\frac{1}{2}}/\log y$, $y\geqq 2$.

A. L. Whiteman.

Wright, E. M. Equal sums of like powers. Proc. Edinburgh Math. Soc. (2) **8**, 138–142 (1949).

The reviewer [Scripta Math. **13**, 37–41 (1948); these Rev. **9**, 78] presented the following theorems. (A) If $\mu_1, \mu_2, \cdots, \mu_n$ are any n integers $(n>1)$ then the sum

$$S(r, n, k)=\sum_{r, n}(\mu_1 a_1+\cdots+\mu_n a_n)^k,$$

extended over all sets of n integers $(a_1, \cdots, a_n)$ for which $a_1+a_2+\cdots+a_n\equiv r \pmod{s}$, $0\leqq a_i<s$, is not dependent upon r for $k=0, 1, 2, \cdots, n-1$. (B) If $\mu_1\cdots\mu_n\neq 0$, then $S(r, n, n)$ depends on r. The proofs were made to depend on formal power series and roots of unity. In the present paper the author gives two very simple proofs of (A) depending only on the multinomial theorem and the factor theorem for polynomials. Use is made of a lemma to the effect that if $\phi(x_1, \cdots, x_n)$ is not really a function of all its variables then the sum $\sum_{r, n}\phi(a_1, \cdots, a_n)$ is independent of r. The author proves (B) via the lemma: The sum $\sum_{r, n}a_1a_2\cdots a_n$ is not independent of r. The proof here is not as immediate as the proofs of (A). By a simple argument the author shows how to construct sets of fewer than s^k numbers having equal sums of like powers up through the kth power.

D. H. Lehmer (Berkeley, Calif.).

Erdös, P. On the coefficients of the cyclotomic polynomial. Portugaliae Math. **8**, 63–71 (1949).

Let $F_n(x)$ be the nth cyclotomic polynomial and let A_n be the absolute value of its numerically largest coefficient. The author proves that if m is a sufficiently large positive integer and n is the product of the first $[m^{1/10}]$ primes not less than m, then the maximum value of $F_n(z)$ on the unit circle is greater than $\exp\{n^{0.6931/(\log\log n)}\}$. Besides number-theoretic arguments the proof uses a theorem on polynomials first stated explicitly by Turán [Acta Univ. Szeged. Sect. Sci. Math. **11**, 106–113 (1946), theorem 3; these Rev. **8**, 266, last theorem quoted] but implicit in a paper of M. Riesz [Jber. Deutsch. Math. Verein. **23**, 354–368 (1914)]. The author's result implies that there is a positive constant

c such that $A_n > \exp\{n^{c/(\log\log n)}\}$ for infinitely many positive integers n. This greatly improves an earlier result of the author [Bull. Amer. Math. Soc. **52**, 179–184 (1946); these Rev. **7**, 242] in the same direction. As the reviewer has remarked [ibid. **55**, 1180–1181 (1949); these Rev. **11**, 329], it is very easy to prove, on the other side, that there is a positive constant C such that $A_n < \exp\{n^{C/(\log\log n)}\}$ for all n greater than 2.

Partial list of errata. On p. 63, l. 16 insert "maximal" between "right" and "order." On p. 65, l. 9 the sign before $x\sum_{2t}$ should be minus. On p. 65, l. 15 replace q by 9. On p. 65, l. 21 the last subscript should be $2t-1$. On p. 66, l. 5 replace $\binom{k}{2t}$ by $\binom{k}{2r}$. On p. 66, l. 8 replace $\sum_r$ by $\sum_{2r}$. On p. 69, l. 18 replace $2t-2$ (the second time it occurs) by $2t-1$.

P. T. Bateman (Urbana, Ill.).

Hasse, Helmut. Osservazioni riguardanti funzioni ellittiche e numeri algebrici. Ann. Mat. Pura Appl. (4) **29**, 225–242 (1949).

Using the congruence groups mod 3 and 4, the author establishes Hurwitz's results on the effect of the modular group $\{S\}$ on the branches of the 12th roots Δ_{12} of the discriminant $g_2^3(W)-27g_3^2(W)$, where

$$W=\begin{pmatrix}W_1\\W_2\end{pmatrix}, \quad \Delta_{12}(SW)=\chi_{12}(S)\Delta_{12}(W).$$

Subsequently, similar results are discussed for quotients of suitably modified elliptic σ-functions. Next, the significance of the nonanalytic function $D(W)=|W|^{\frac{1}{2}}\Delta_{12}(W)^{\frac{1}{2}}\bar{\Delta}_{12}(W)^{\frac{1}{2}}$, and of analogous "normal" elliptic functions, as invariants of ray classes and for the determination of class numbers and units (with generalizations to real quadratic fields) is indicated. In this connection see Dedekind [J. Reine Angew. Math. **121**, 40–123 (1900)].

O. F. G. Schilling.

van der Corput, J. G. La deuxième perle de la théorie des nombres. Math. Centrum Amsterdam, Scriptum no. **4**, i+14 pp. (1950).

This is an expository article on the (α, β)-theorem of additive number theory. In particular, a proof of the author's theorem 2 is given [cf. Nederl. Akad. Wetensch. Proc. **50**, 252–261 = Indagationes Math. **9**, 159–168 (1947); these Rev. **8**, 566].

P. Scherk (Saskatoon, Sask.).

Khintchine, A. Ya. On some applications of the method of the additional variable. Amer. Math. Soc. Translation no. **18**, 14 pp. (1950).

Translated from Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 188–200 (1948); these Rev. **10**, 512.

ANALYSIS

* **Jeffreys, Harold, and Jeffreys, Bertha Swirles. Methods of Mathematical Physics. 2d ed.** Cambridge, at the University Press, 1950. xi+708 pp. $15.00.

The first edition [1946] was reviewed in these Rev. **8**, 447. The general plan of the book remains the same, but there are numerous additions and improvements of details.

R. P. Boas, Jr. (Evanston, Ill.).

Ryll-Nardzewski, C. Sur les moyennes. Studia Math. **11**, 31–37 (1949).

Stemming from considerations of the classical Kolmogoroff-Nagumo theorem (1930), several characterizations of functions of the form (1) $M(x, y)=f^{-1}[(f(x)+f(y))/2]$, where $f(x)$ is continuous and strictly increasing, have been given. The author now shows that a function $M(x, y)$ satisfies (1) if and only if $M(x, y)$ is an increasing function of x and of y, is continuous on the line $x=0$, and satisfies $M[x, M(y, z)]=M[M(x, y), M(z, x)]$ for all real (x, y, z).

E. F. Beckenbach (Los Angeles, Calif.).

Sanz Sanchez, Andres. Study of a general mean. Gaceta Mat. (1) **1**, 183–187 (1949). (Spanish)

On the basis of his knowledge of the definition and fundamental properties of the mean of order t, $-\infty \leqq t \leqq \infty$, for a finite set of positive values $x_1, \cdots, x_n$, the author develops the known analogous definition and properties of the mean of order t for a continuous positive function $\phi(x)$.

E. F. Beckenbach (Los Angeles, Calif.).

Thielman, H. P. On generalized means. Proc. Iowa Acad. Sci. **56**, 241–247 (1949).

The author gives an elementary exposition, on a postulational basis, of the notion of mean value function. He gives examples of functions having, and of functions not having, mean value properties of symmetry, bisymmetry, homogeneity, and so on, and points out how generalized mean value functions are used to define generalized convex functions.

E. F. Beckenbach (Los Angeles, Calif.).

Aczél, Jean. Inégalités. Gaz. Mat., Lisboa **10**, no. 39, 5–7 (1949); no. 40, 5–9 (1949); nos. 41–42, 4–11 (1949); **11**, no. 43, 10 (1950).

In the first four of the five chapters of the present tract, the author discusses the elements of the known classical theory of continuous convex functions, Jensen's inequality, other characteristic properties of convex functions, and convex functions of several variables, respectively. Simple but interesting relevant analytical and geometrical problems and exercises are stated in these chapters, the solutions appearing in the next succeeding chapters.

E. F. Beckenbach (Los Angeles, Calif.).

Pleijel, Arne. An inequality. Mat. Tidsskr. A. **1949**, 67–69 (1949). (Swedish)

Let $f(x)$ be real, of period 2π, with a second derivative of L^2. Let all integrals be over $(0, 2\pi)$. By using Fourier series the author obtains sharp upper and lower bounds for the difference $2\pi\int f^2-(\int f)^2$, namely $2\pi\int(f')^2$ and $2\pi\{\int(f')^2\}/\int(f'')^2$, respectively. The upper bound is essentially "Wirtinger's inequality" [see Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, p. 185, theorem 258].

R. P. Boas, Jr. (Evanston, Ill.).

Vythoulkas, D. Generalization of the Schwarz inequality. Bull. Soc. Math. Grèce **24**, 119–127 (1949). (English. Greek summary)

Given an ordered set of k positive numbers $x_{1,0} \leqq \cdots \leqq x_{k,0}$, the author constructs a second set of k positive numbers by taking the geometric mean of each $x_{j,0}$ with its cyclic successor: $x_{j,1}=(x_{j,0}x_{j+1,0})^{\frac{1}{2}}$, $j=1, \cdots, k$, where $x_{k+1,0}=x_{1,0}$. Arranging the $x_{j,1}$ in order of magnitude, and repeating the process indefinitely, he obtains k infinite sequences $\{x_{j,n}\}$, $j=1, \cdots, k$; $n=1, 2, \cdots$. It is shown that

$$\lim_{n\to\infty} x_{j,n}=(x_{1,0}\cdots x_{k,0})^{1/k}$$

for each j. Thus we have an approximate method for com-

puting kth roots by means of square roots. The author points out that the present method yields an approximate geometrical solution of the problem of duplicating the cube. Applying the method analytically, he derives a special case of Hölder's inequality; as the title of the paper suggests, his result is also a generalization of the Schwarz inequality.
E. F. Beckenbach (Los Angeles, Calif.).

Popović, V. Une démonstration de l'inégalité de Cauchy. Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 133–135 (1949). (Serbian. French summary)
An inductive proof of the inequality between the arithmetic and geometric mean. *R. P. Boas, Jr.*

Sunyer i Balaguer, Ferran. Sur des résultats de M. S. Mandelbrojt. C. R. Acad. Sci. Paris **231**, 18–20 (1950).
The author generalizes Mandelbrojt's "fundamental inequality" [Ann. Sci. École Norm. Sup. (3) **63**, 351–378 (1947); these Rev. **9**, 229, 735] and deduces a new result on the representation of a function by a sequence of Dirichlet polynomials $\sum_1^n a_k^{(n)} e^{-\lambda_k s}$. The detailed statements are too long to reproduce here. *R. P. Boas, Jr.*

Timan, A. F. Quasi-smooth functions. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 128–130 (1950). (Russian)
The results of the note have already been stated elsewhere [Doklady Akad. Nauk SSSR (N.S.) **70**, 961–963 (1950); these Rev. **11**, 422]. *A. Zygmund.*

Obrechkoff, N. Sur le comportement asymptotique des dérivées des fonctions. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 2–3, 5–8 (1949).
Let $\varphi(x)$ be a real function such that, as $x\to\infty$, $\varphi(\lambda x)/\varphi(x)\to\lambda^m$ for each positive λ. The principal theorems are as follows. (1) If, for large x, $f(x)\sim A\varphi(x)$, $f^{(n)}(x)>-Mx^{-n}\varphi(x)$, and $\varphi(x)$ is monotonic, then for $1\leqq i\leqq n-1$ we have $f^{(i)}(x)\sim Am(m-1)\cdots(m-i+1)x^{-i}\varphi(x)$. (2) If $f^{(n)}(x)$ is monotonic for large x and $f(x)\sim A\varphi(x)$ then $f^{(n)}(x)\sim Am(m-1)\cdots(m-n+1)x^{-n}\varphi(x)$. Various consequences are deduced and analogous results are given for sequences. *R. P. Boas, Jr.* (Evanston, Ill.).

Hopf, Eberhard. On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature. Proc. Amer. Math. Soc. **1**, 80–85 (1950).
This paper gives a complete proof for the following theorem of S. Bernstein (whose proof contains a gap). Let $f(x, y)$ be of class C'' and defined for all x and y, and moreover satisfy $\Delta=f_{xx}f_{yy}-f_{xy}^2\geqq 0$, $\Delta\not\equiv 0$; then $f(x, y)$ cannot be $o(r)$ for large $r=(x^2+y^2)^{\frac{1}{2}}$. *H. Busemann.*

Mickle, Earl J. A remark on a theorem of Serge Bernstein. Proc. Amer. Math. Soc. **1**, 86–89 (1950).
Under the same hypothesis as in the preceding review it is shown that $f(x, y)$ cannot be bounded. This is the form in which Bernstein gave his theorem originally [Math. Z. **26**, 551–558 (1927)]; he later [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 285–290 (1942); these Rev. **5**, 14] gave the stronger form of the preceding review (with the same gap in the proof).
H. Busemann (Los Angeles, Calif.).

Kneschke, A. Zur Theorie der Interpolation. Math. Z. **52**, 137–149 (1949).
Aus dem Newtonschen Interpolationstheorem [G. Kowalewski, Interpolation und genäherte Quadratur, Teubner, Leipzig-Berlin, 1932, s. 74]

$$f(x)=\sum_{\mu=1}^{n} f_\mu(x_\mu)L_\mu(x)+\int_\alpha^\beta K(x,\xi)f^{(n)}(\xi)d\xi,$$

$\alpha\leqq x\leqq\beta$, $\alpha\leqq x_1<\cdots<x_n\leqq\beta$ mit

$$K(x,\xi)=\tfrac{1}{2}\{(x-\xi)^{n-1}/(n-1)!\}\operatorname{sgn}(x-\xi)-\sum_{\mu=1}^{n}L_\mu(x)\{(x_\mu-\xi)^{n-1}/(n-1)!\}\operatorname{sgn}(x_\mu-\xi),$$

$$L_\mu(x)=L(x)/(x-x_\mu)L^1(x_\mu),\quad L(x)=\prod_{\mu=1}^{n}(x-x_\mu),$$

folgt durch ρ-fache Ableitung nach x:

$$f^{(\rho)}(x)=\sum_{\mu=1}^{n}f(x_\mu)L_\mu^{(\rho)}(x)+\int_\alpha^\beta K^{(\rho)}(x,\xi)f^{(n)}(\xi)d\xi;\qquad \rho=0,\cdots,n-1.$$

Ersetzt man die Ableitungen von f in der Restfunktion schrittweise durch analoge Entwicklungen, unter gleichseitiger Vertauschung der Integrationen in den Doppelintegralen, so gelangt man zu dem Kowalewskischen Interpolationstheorem

$$(1)\quad f^{(\rho)}(x)=\sum_{k=0}^{\nu-1}\sum_{\mu=1}^{n}f^{(k(n-\rho))}(x_\mu)I_{\mu k}(x)+\int_\alpha^\beta K_{\nu-1}^{(\rho)}(x,\xi)f^{(\nu(n-\rho)+\nu)}(\xi)d\xi$$

mit den Kerniterationen:

$$K_i^{(\rho)}(x,\xi)=\int_\alpha^\beta K_{\nu-1}^{(\rho)}(x,\eta)\cdot K^{(\rho)}(\eta,\xi)d\eta=\int_\alpha^\beta K^{(\rho)}(x,\eta)\cdot K_{\nu-1}^{(\rho)}(\eta,\xi)d\eta,$$

$$K_0^{(\rho)}(x,\xi)=K^{(\rho)}(x,\xi).$$

$$I_{\mu 0}(x)=L_\mu^{(\rho)}(x);\quad I_{\mu k}(x)=\int_\alpha^\beta K_{\nu-1}^{(\rho)}(x,\xi)\cdot L_\mu^{(\rho)}(\xi)d\xi,\quad k\geqq 1.$$

Sonderfälle: $\rho=0$, $n=2$, $x_1=\alpha$, $x_2=\beta$ (Euler-Bernoullische Interpolationstheorem); $\rho=n-1$, $n=2$, $x_1=\alpha$, $x_2=\beta$ (Euler-Maclaurinsche Interpolationstheorem); $\rho=0$, $n=1$ (Taylorsche Interpolationsformel mit den Bernoullischen Integralrestglied). Aus (1) werden zwei allgemeine Quadraturformeln abgeleitet. *S. C. van Veen* (Delft).

Kneschke, A. Theorie der genäherten Quadratur. J. Reine Angew. Math. **187**, 115–128 (1949).
Das Ziel dieser Untersuchungen ist, die genäherte Quadratur ganz und gar aus der Interpolationstheorie herauszulösen, und eine selbstständige Theorie der genäherten Quadratur zu entwicklen. An der Spitze tritt das folgende Grundtheorem. Die für jede stetige und $(m+1)$-mal stetig differenzierbare Funktion $f(x)$ geforderte Quadraturformel

$$\int_{x_0}^{x_n}f(x)dx=\sum_{\rho=0}^{n}\sum_{\sigma=0}^{k_\rho}A_{\rho\sigma}f^{(\sigma)}(x_\rho)+\int_{x_0}^{x_n}f^{(m+1)}(\xi)Q(\xi)d\xi$$

$(k_\rho\leqq m\leqq n+\sum_{\rho=0}^{n}k_\rho)$ lässt sich durch ein von $f(x)$ unabhängiges System von Koeffizienten $A_{\rho\sigma}$ erfüllen, die der Identität

$$\sum_{\rho=0}^{n}\sum_{\sigma=0}^{k_\rho}A_{\rho\sigma}\{(x_\rho-\xi)^{m-\sigma}/(m-\sigma)!\}=\{(x_n-\xi)^{m+1}-(x_0-\xi)^{m+1}\}/(m+1)!$$

und $n-m+\sum_{\rho=0}^{n}k_\rho$ frei wählbaren, von einander unabhängigen, linearen Nebenbedingungen genügen. Die erzeu-

gende Funktion $Q(\xi)$ ist in ihrer Abhängigkeit von $A_{\rho\sigma}$ durch

$$Q(\xi)=\tfrac{1}{2}\left\{\frac{(x_0-\xi)^{m+1}+(x_n-\xi)^{m+1}}{(m+1)!}-\sum_{\rho=0}^{n}\sum_{\sigma=0}^{k_\rho}A_{\rho\sigma}\frac{(x_\rho-\xi)^{m-\sigma}}{(m-\sigma)!}\operatorname{sgn}(x_\rho-\xi)\right\}$$

gegeben. Dann bestehen die folgenden Beziehungen: $\sum_{\rho=0}^{n}\sum_{\sigma=0}^{k_\rho}A_{\rho\sigma}f^{(\sigma)}(x_\rho)\sim$Newtonsche Interpolationssumme; $A_{\rho\sigma}\sim$Newtonsche Faktoren; $Q(\xi)\sim$Newtonsche Funktion. Für $m=M=n+\sum_{\rho=0}^{n}k_\rho$ sind die Newtonsche Faktoren eindeutig bestimmt. Die Betrachtungen werden erweitert auf Funktionen zweier Variablen. Schliesslich werden einige Sonderfälle behandelt. *S. C. van Veen* (Delft).

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, Wacław. Sur les séries infinies de nombres ordinaux. Fund. Math. **36**, 248–253 (1949).

The author studies rearrangements of infinite series $\alpha_1+\cdots+\alpha_n+\cdots$ in the usual sense, but takes the terms to be order types or ordinal numbers. He notes first that, if η denotes the order type of the rational numbers in their natural order, and $\alpha_n=n\eta$ for every natural number n, then no two rearrangements yield the same sum. If, however, every term of the series is an ordinal number, only a finite number of different sums can be obtained by rearranging the series. (In the course of the proof it is shown that, if ρ is an ordinal number such that $\xi+\rho=\rho$ for every $\xi<\rho$, then, for any ordinal number α, $\xi+\alpha+\rho=\alpha+\rho$ for every $\xi<\rho$.) In particular, if every term is such that there are infinitely many other terms which are not less than it, then the sum of the series of ordinals is independent of the order of its terms; but the series $\omega+1+2+\cdots+n+\cdots$ shows that the converse is false. If $\alpha_i=\omega$ for $i=1, 2, \cdots, n-1$, $\alpha_n=\omega^2$, and $\alpha_i=1$ for $i\geqq n+1$, then rearrangement produces precisely n different sums, viz., $\omega^2+\omega\cdot k$ $(k=1, \cdots, n)$. The author makes a few remarks on the case in which the indices of the terms of the original series form a well-ordered set of type $\tau>\omega$; for the case in which $\tau<\omega$ (the author considers only $\tau=2$ and $\tau=3$), see the two following reviews.

F. Bagemihl (Rochester, N. Y.).

Wakulicz, A. Sur la somme d'un nombre fini de nombres ordinaux. Fund. Math. **36**, 254–266 (1949).

Denote by m_n the maximum number of different ordinals one can obtain by adding, in all possible ways, n ordinals. The author proves that for $n>20$

$$(1)\qquad m_n=81^{6[(n-1)/5]-n+1}\cdot 193^{n-1-5[(n-1)/5]}.$$

The author also determines the value of m_n for $n<20$. He also investigates the following question. Let $k\leqq m_n$. Can one find n ordinal numbers so that one can obtain, by adding them in all possible ways, precisely k summands? He shows that this can be done for $n\leqq 4$ but not for $n=5$. [Reviewer's note. There is a slight numerical error on p. 259. The author states that $f(4, 5, 5, 5, 5)<f(6, 6, 6, 6, 6)$, but this is incorrect since $33\cdot 81^4>193^4$. The correct formula for numbers of the form $5x+5$ is given by $m_{5x+5}=33\cdot 81^x$. The reviewer [see the following review] obtained the correct formula for m_n almost simultaneously.] *P. Erdös* (Aberdeen).

Erdös, P. Some remarks on set theory. Proc. Amer. Math. Soc. **1**, 127–141 (1950).

This paper contains a number of unrelated results in set theory, among which are the following. (I) If n is a natural number, let $f(n)$ denote the maximum number of distinct values that a sum of n ordinals can assume if one permutes the terms in all possible ways. Then

$$f(n)=\max_{k\leqq n-1}(k2^{k-1}+1)f(n-k).$$

The values of $f(2), \cdots, f(15)$ are 2, 5, 13, 33, 81, 193, 449, 1089, 2673, 6561, 15633, 37249, 88209, 216153; for $x\geqq 3$, $f(5x+1)=81^x$, $f(5x+2)=193\cdot 81^{x-1}$, $f(5x+3)=193^2 81^{x-1}$, $f(5x+4)=193^3 81^{x-1}$, $f(5x+5)=33\cdot 81^x$; and for $n\geqq 21$, $f(n)=81f(n-5)$. A. Wakulicz [see the preceding review] has treated the same problem by calculating $f(1), \cdots, f(20)$ and giving a formula for $f(n)$ for $n>20$. (II) Let X be a set of power $m\geqq\aleph_0$, and let A, B denote sets whose elements are subsets of X. Then A and B are said to be α-orthogonal, if the intersection of any element of A with any element of B is a set of power less than $\aleph_\alpha\leqq m$. The α-orthogonal sets A and B are called α-complete, if no subset of X can be added to A or to B without destroying the α-orthogonality. (Definitions of this sort were first given by N. Lusin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 175–178 (1943); these Rev. 6, 42].) Then the number of α-complete orthogonal pairs is 2^{m^γ}, where $\gamma=\aleph_\alpha$. The proof of this makes use of the generalized hypothesis of the continuum. (III) Let S, with $|S|\geqq\aleph_0$, be any subset of k-dimensional Euclidean space. Then there exists a subset S_1 of S, with $|S_1|=|S|$, such that the distance between any two points of S_1 is different from that between any other such pair. (IV) The following generalizes a result of König [Theorie der endlichen und unendlichen Graphen . . . , Akademische Verlagsgesellschaft, Leipzig, 1936, pp. 220–223]: If G is a graph of order $m\geqq\aleph_0$, where every vertex is connected by an edge to each of at least m different vertices, then G is the product of linear factors. (V) Let the set S have power $m\geqq\aleph_0$, let $n<m$, and suppose that to every $a\in S$ there corresponds a subset $f(a)$ of S such that $f(a)$ has power less than n and $a\overline{\in}f(a)$. The elements a and b of S are called independent, if $a\overline{\in}f(b)$ and $b\overline{\in}f(a)$, and a subset S' of S is called independent, if every two elements of S' are independent. Does there exist an independent set of power m? S. Piccard [Fund. Math. **29**, 5–8 (1937); C. R. Soc. Sci. Varsovie **30**, 12–18 (1937)] has shown that, under certain conditions on m, the answer to this question raised by Sierpiński and Ruziewicz is affirmative, and the author now shows that, assuming the generalized hypothesis of the continuum, the answer is always affirmative. In addition to other results related to those just mentioned, the paper contains theorems dealing with Hamel bases, measure, and ordinal numbers.

F. Bagemihl (Rochester, N. Y.).

Bagemihl, F. A theorem on infinite products of transfinite cardinal numbers (correction). Quart. J. Math., Oxford Ser. (2) **1**, 160 (1950).

Cf. the same J., Oxford Ser. (1) **19**, 200–203 (1948); these Rev. **10**, 359.

Fraïssé, Roland. Sur les types de polyrelations et sur une hypothèse d'origine logistique. C. R. Acad. Sci. Paris **230**, 1557–1559 (1950).

The author announces more results belonging to a subject considered in two earlier notes [C. R. Acad. Sci. Paris **228**, 1682–1684 (1949); **230**, 1022–1024 (1950); these Rev. **11**, 17, 585]. For reasons of space the concepts and propositions cannot be reproduced here. The title alludes to a conjecture, formulated in terms of polyrelations, which implies that one can "state a law which assigns to each countable ordinal

w a well-ordering of the whole numbers which has precisely the ordinal w." *R. Arens* (Los Angeles, Calif.).

Kalmár, László. Über die Cantorsche Theorie der reellen Zahlen. Publ. Math. Debrecen 1, 150–159 (1950).

This is a presentation of the classical Cantor construction which is very close to that of van der Waerden [Moderne Algebra, vol. I, 2d ed., Springer, Berlin, 1937] but exhibits still stronger algebraic tendencies. The field of real numbers is defined as the quotient of the ring C of Cauchy sequences of rational numbers by the ideal I of sequences having 0 as a limit (van der Waerden's definition); the difference lies in the definition of Cauchy sequences, which are defined as those which are such that any subsequence lies in the same class mod I. *J. Dieudonné* (Nancy).

***Carathéodory, C. Masstheorie und Integral.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 195–208, Rome, 1943.

An outline of the theory of measure and integration on a Boolean algebra. *P. R. Halmos* (Chicago, Ill.).

Banach, S. Sur la mesure dans les corps indépendants. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 71–90 (1947). (Ukrainian and French. Russian summary)

A proof (substantially the same as the author's original proof) of the author's theorem on the existence of measures on the Boolean algebra generated by a countably independent class of Boolean algebras [Studia Math. **10**, 159–177 (1948); these Rev. **10**, 600]. A much simpler proof of the theorem has been published recently by Sherman [see the following review]. *P. R. Halmos* (Chicago, Ill.).

Sherman, S. On denumerably independent families of Borel fields. Amer. J. Math. **72**, 612–614 (1950).

The author gives a very short and simple proof of the theorem of Banach stated in the preceding review.

P. R. Halmos (Chicago, Ill.).

Kuipers, L., and Meulenbeld, B. New results in the theory of C-uniform distribution. Nederl. Akad. Wetensch., Proc. **53**, 822–827 = Indagationes Math. **12**, 266–271 (1950).

The authors continue their work on C-uniform distribution. Among others they prove, sharpening previous results, the following theorem. Let F be a sequence of intervals $Q: 0 \leqq t < T$ with $T \to \infty$. If $f(t)$ is differentiable and satisfies $|tf'(t)| < k$ for $t > t_0 > 0$, then $f(t)$ is not C-uniformly distributed (mod t) in the intervals Q of F. *P. Erdös.*

Menger, Karl. What paths have length? Fund. Math. **36**, 109–118 (1949).

To each ordered pair of points x, y of the closed interval $[0, 1]$ a real number $\delta(x, y)$ is attached with the sole restriction that if x, y have a sufficiently small Euclidean distance, then $|\delta(x, y)|$ is arbitrarily small. Calling the number $\delta(x, y)$ the distance from x to y, the author formulates necessary and sufficient conditions that the interval $[0, 1]$ so distancialized shall have a finite length. *L. M. Blumenthal.*

Verčenko, I. Ya. Investigations on the theory of area of surfaces. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 205–207 (1950). (Russian)

In the first part the author states (without any details) that, for surfaces of the form $z = w(x, y)$ of bounded area, compactness can be achieved by, and only by, admitting, instead of the traditional continuous surfaces, more general "measurable surfaces." The remainder of the abstract repeats the substance of a previous note [Doklady Akad. Nauk SSSR (N.S.) **68**, 5–8 (1949); these Rev. **11**, 240].

L. C. Young (Madison, Wis.).

Theory of Functions of Complex Variables

Lohin, I. F. On a representation of an entire analytic function of the first order of normal type. Doklady Akad. Nauk SSSR (N.S.) **72**, 629–632 (1950). (Russian)

The author is interested in expansions of an entire function of exponential type in the form $F(z) = \sum A_n P_n(z)$, where $2\pi i A_n = \int_C [u(\zeta)]^n f(\zeta)\, d\zeta$, $f(\zeta)$ is the Borel transform of $F(z)$, C is an appropriate contour, and $P_n(z)$ are suitable polynomials; from such expansions he derives uniqueness theorems. His results do not appear to differ essentially from those of R. C. Buck [Proc. Nat. Acad. Sci. U. S. A. **33**, 288–292 (1947); Duke Math. J. **15**, 879–891 (1948); Trans. Amer. Math. Soc. **64**, 283–298 (1948); these Rev. **9**, 232; **10**, 693]. *R. P. Boas, Jr.* (Evanston, Ill.).

Uhl, Wolfgang. Über die Darstellung ganzer Funktionen mittels der Stirling'schen Reihe bei Hermite'scher Interpolation. Mitt. Math. Sem. Univ. Giessen **33**, ii+42 pp. (1944).

In this thesis the author seeks conditions for the representability of an entire function by the generalized Stirling series which use the values of the function and its first derivative at the positive and negative integers. For the ordinary Stirling series, Nörlund [Leçons sur les séries d'interpolation, Gauthier-Villars, Paris, 1926] proved that the appropriate class consists of entire functions of exponential type whose indicator functions do not exceed a certain function $\psi(\theta)$ (together with additional finer restrictions on their growth). [For the definition of $\psi(\theta)$ see Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Springer, Berlin, 1925, vol. 1, section III, no. 115.] The author's main result is that the "Hermite-Stirling series" $\psi(\theta)$ may be replaced, roughly speaking, by $2\psi(\theta)$. More precisely, it is sufficient that, for large r, $\log |F(re^{i\theta})| < \beta \log r + 2r\psi(\theta)$, $\beta < 0$; necessary, that for large r,

$$\log |F(re^{i\theta})| < (11/3) \log r + 2r\psi(\theta).$$

R. P. Boas, Jr. (Evanston, Ill.).

Ilieff, Ljubomir. Über die Nullstellen einer Klasse von ganzen Funktionen. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 2–3, 9–11 (1949).

Let $f(t)$ be nonnegative, increasing, and integrable in $(0, 1)$; let $p(z)$ be a polynomial whose zeros are in $\alpha \leqq \Re(z) \leqq \beta$, or an entire function which is the limit of such polynomials. Let $f_1(t) = \int_t^1 f(u)\, du$. Then $\int_0^1 f_1(t)\, |p'(z+t) + p'(z-t)|\, dt$ has all its zeros in $\alpha \leqq \Re(z) \leqq \beta$. This includes the author's previous results [same C. R. **1**, no. 2–3, 15–18 (1948); **2**, no. 1, 17–20 (1949); these Rev. **10**, 530; **11**, 236].

R. P. Boas, Jr. (Evanston, Ill.).

Straus, E. G. On entire functions with algebraic derivatives at certain algebraic points. Ann. of Math. (2) **52**, 188–198 (1950).

Let f be an entire function, and suppose that $f^{(n)}(z)$ is integral for each z in a finite set S, and for all $n = 0, 1, 2, \cdots$. The problem posed by the author is that of finding ρ and σ such

that if f is of growth less than order ρ type σ, f must be a polynomial. For $S=\{0, 1, 2, \cdots, k\}$ he finds $\rho=k+1$, $\sigma=(k!)^{-2}$. For $S=\{0, p_1/q, p_2/q, \cdots, p_k/q\}$ with $(p_1, p_2, \cdots, p_k/q)=1$, $\rho=k+1$ and $\sigma=q^k/m^2$ where m depends upon the numbers $p_1, \cdots, p_k$. For $S=\{0, r\}$ with r rational, he obtains the sharper estimate $\rho=2$, $\sigma=r^{-1}$. These are generalized as follows. Let K be an algebraic field of degree r, and suppose that $S=\{0, \alpha_1, \cdots, \alpha_{k-1}\}$ where the α_i are chosen from K and obey certain conditions. Suppose that (rational) integers q_n exist such that $q_n f^{(n)}(z)$ is an integer in K for each $z \epsilon S$ and for $n=0, 1, 2, \cdots$. Let $|\overline{f^{(n)}(z)}|=O((A+\epsilon)^n n^{tn})$ and $q_n=O((B+\epsilon)^n n^{sn})$, where $|\overline{\alpha}|$ denotes the maximum of the absolute values of the conjugates of α. Then, there exist ρ_0 and σ_0 (explicitly determined by A, B, s, and t) such that if f is of growth less than order ρ_0 type σ_0, f must be a polynomial. The proof is based on a lemma which asserts that if the entire function $g(z)=\sum \alpha_n z^n/n!$ is of sufficiently slow growth, and if the α_n are in K, and also limited in growth, then g must be a polynomial. The author's main theorem may be applied to give transcendence proofs. That of π and that of e^α for algebraic α result from the choice of $f(z)$ as in $\sin z$, and e^z. Additional examples may be obtained by choosing f to satisfy specified differential equations or difference equations, having algebraic coefficients.

R. C. Buck (Madison, Wis.).

Shah, S. M. The maximum term of an entire series. IV. Quart. J. Math., Oxford Ser. (2) **1**, 112–116 (1950).

[For parts III and V, cf. the same J., Oxford Ser. (1) **19**, 220–223 (1948); J. Indian Math. Soc. (N.S.) **13**, 60–64 (1949); these Rev. **10**, 289; **11**, 508.] Let $f(z)$ be an entire function of infinite order, $\mu(r)$ the maximum term of its power series, $\nu(r)$ the rank of this term, $M(r)$ the maximum modulus of $f(z)$, $L(r)$ a logarithmico-exponential function tending to ∞ with r. The author first proves that $\liminf \{\log M(r)\}/\nu(r)=0$, which he compares with the known results (for finite order) $\log M(r) \sim \log \mu(r)$ and $\liminf \log \mu(r)/\nu(r)=0$. Then he shows that $\nu(r)$ cannot be replaced by $\nu(r)/\varphi(r)$ with $\varphi(r) \uparrow \infty$. Finally he gives relations connecting $\limsup$ and $\liminf$ of $\{\log \log \mu(r)\}/L(r)$ with those of $\log \mu(r) \cdot rL'(r)/\nu(r)$. *R. P. Boas, Jr.*

Nazim Terzioğlu, A. Über den Koebeschen Verzerrungssatz. Rev. Fac. Sci. Univ. Istanbul (A) **15**, 113–118 (1950). (German. Turkish summary)

The Koebe theorem states that if the analytic function $f(z)$ is regular and schlicht in $|z|<1$, $f(0)=0$ and $f'(0)=1$, then $w=f(z)$ maps $|z|<1$ onto a domain D_w which covers the circle $|w|<\frac{1}{4}$. The author gives two new proofs of this theorem. First D_w is mapped by an explicit transformation $s=s(w)$ onto a domain D_s which contains in its interior the segment $0 \leqq s<1$ but has $s=1$ as a boundary point, and such that if s is interior to D_s, then $1/s$ is exterior to D_s. Let $\omega(\rho e^{i\varphi})=\log |s(w(\rho e^{i\varphi}))/\rho|$. Then

$$\omega(\rho e^{i\varphi})=\lambda+\sum_{n=1}^{\infty}(\alpha_n \cos n\varphi+\beta_n \sin n\varphi)\rho^n$$

and we set

$$(*) \qquad [m(x)]^2=2\pi(x+\lambda)^2+\pi\sum_{n=1}^{\infty}(\alpha_n^2+\beta_n^2)e^{2nx},$$

where $\lambda=-\log 4d$ and d is the shortest distance from $w=0$ to the boundary of D_w. The function $m(x)$ is next related to an area using strip mappings, from which it is deduced that $m(x)m'(x) \leqq 0$ for all $x \leqq 0$. From (*),

$$m(0)m'(0)=2\pi\lambda+\pi\sum_{n=1}^{\infty}n(\alpha_n^2+\beta_n^2) \leqq 0$$

which implies $\lambda \leqq 0$, giving the first proof of Koebe's theorem. The second proof is completed by showing that $-2\pi(x+\lambda) \geqq m(0)m'(0)-m(x)m'(x) \geqq -2\pi x$.

G. Springer (Cambridge, Mass.).

Wolibner, W. Sur les coefficients des fonctions analytiques univalentes à l'extérieur d'un cercle. Studia Math. **11**, 126–132 (1949).

Let Φ be the class of schlicht analytic functions in $|z|>1$ which are normalized by the expansion

$$f(z)=z+\sum_{n=1}^{\infty}b_n/z^n.$$

Let $B_n=\max |b_n|$ for $f \epsilon \Phi$. It has been conjectured that $B_n=2/(n+1)$ with verification in the cases $n=1$ and $n=2$ [Schiffer, Bull. Soc. Math. France **66**, 48–55 (1938)]. The author now considers subclasses of Φ and shows that for these subclasses $B_n=2/(n+1)$, adding strength to the conjecture. In particular, he shows that if $f(z)$ belongs to Φ and in addition satisfies one of the two hypotheses in group I and one of the two hypotheses in group II stated below, then $|b_n| \leqq 2/(n+1)$. Group I: (IA) $f(z) \neq 0$ in $|z|>1$; (IB) n is an odd number. Group II: (IIA) $b_1=b_2=\cdots=b_{[\frac{1}{2}(n-1)]}=0$, where $[\frac{1}{2}(n-1)]$ is the integer part of $\frac{1}{2}(n-1)$; (IIB) let p and r be integers, $p \geqq 0$, and d be the greatest common divisor of p and $r+1$, and let $n=Sp+r$, $r \neq -1$, $r>-p$, and $1 \leqq S \leqq (p+r+1)/d$, and finally suppose that $f(z)$ can be expanded in $|z|>1$ in a series of the form

$$f(z)=z+\sum_{s=1}^{S}b_{sp+r}/z^{sp+r}+\sum_{k=n+1}^{\infty}b_k/z^k.$$

As an immediate consequence of this theorem, he concludes that if we have a sequence of functions $\{f_p(z)\}$, $p=1, 2, \cdots$, such that $f_p \epsilon \Phi$ and satisfies (IA), and moreover in $|z|>1$ has the expansion

$$f_p(z)=z+\sum_{s=1}^{\infty}b_s^{(p)}/z^{sp+r}, \qquad r \neq -1,$$

we have $\limsup_{p \to \infty} |pb_s^{(p)}| \leqq 2/s$. He then shows that this result for the excluded case $r=-1$ implies the Bieberbach conjecture, i.e., for regular schlicht functions

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

in $|z|<1$, $|a_n| \leqq n$. *G. Springer* (Cambridge, Mass.).

Džrbašyan, M. M. Metric theorems on completeness and on the representation of analytic functions. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 194–198 (1950). (Russian)

Summary of a thesis. Some of the theorems and related results have been reviewed before [Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 555–568 (1948); Doklady Akad. Nauk SSSR (N.S.) **62**, 581–584 (1948); **66**, 1037–1040 (1949); **67**, 15–18 (1949); these Rev. **10**, 444, 364; **11**, 94, 95]. Let D be a simply connected domain lying in a simply connected infinite domain obtained by removing a finite number of nonoverlapping sectors of opening π/a_i ($\frac{1}{2}<a_i<\infty$, $i=1, 2, \cdots, m$) from the z-plane. Let $h(r)$ be the linear measure of the intersection of D with $|z|=r$. Then $\{z^n\}$ is

complete in the Hilbert space of functions $f(z)$ regular in D and of norm $\|f\| = \iint_D |f(z)|^2 dx\,dy$, if

$$\int r^{-1-a} \log h(r) dr = -\infty,$$

$a = \max_i a_i$.

If $f(z)$ is regular in $|z|<1$ and

$$\iint_{|z|<1} (1-|z|^2)^\alpha |f(z)|^p dx\,dy < \infty$$

$(\alpha > -1, p > 0)$, then (1) $f(z) =$

$$(\alpha+1)\pi^{-1} \int_0^1 \int_0^{2\pi} (1-\rho^2)^\alpha (1-z\rho e^{-i\theta})^{-2-\alpha} f(\rho e^{i\theta}) \rho d\rho d\theta = I(f).$$

If $f(z)$ is meromorphic in $|z|<1$ and if the characteristic function of $f(z)$ satisfies $\int_0^1 (1-\rho)^\alpha T(\rho) d\rho < \infty$, $\alpha > -1$, then $f(z) = Az^k (\pi(z; a_\nu)/\pi(z; b_\nu)) \exp\{2I(\log|f|)\}$ where I is defined by (1) and

$$\pi(z; a_\nu) = \prod_{\nu=1}^{\infty} (1-z/a_\nu) \exp\{2I(\log|1-z/a_\nu|)\}.$$

W. H. J. Fuchs (Ithaca, N. Y.).

Thiem, Le Van. Sur un problème d'inversion dans la théorie des fonctions méromorphes. Ann. Sci. École Norm. Sup. (3) **67**, 51–98 (1950).

This paper is a somewhat more detailed treatment of the material presented in condensed form in a previous publication by the same author [Comment. Math. Helv. **23**, 26–49 (1949); these Rev. **11**, 22]. Concerning an exceptional case not handled in the previous paper, he adds the theorem: If all the defects $\delta_\nu = 0$ and if all the indices ϵ_ν are rational and less than one, the problem of inversion can be solved by an elliptic function. *G. Springer.*

Valiron, Georges. Surfaces de Riemann simplement connexes dont des points de ramification sont donnés. Ann. Mat. Pura Appl. (4) **29**, 321–326 (1949).

The problem is to find a simply connected Riemann surface over the Z-plane such that the projections of the algebraic branch-points comprise an arbitrary sequence of points A_k. If $F(z)$ is a fixed entire function with infinitely many simple zeros a_p it is found that

$$Z = f(z) = F(z)^2 \sum_1^\infty A_k [(z-a_p)^{-2} F'(a_p)^{-2} - (z-a_p)^{-1} F''(a_p) F'(a_p)^{-3}], \qquad p = p_k,$$

maps the z-plane onto a surface with this property provided that the sequence p_k is chosen so that the series

$$\sum_1^\infty |A_k| \cdot |F'(a_p)|^{-1}, \quad \sum_1^\infty |A_k| \, |F''(a_p)| \, |F'(a_p)|^{-3}$$

converge. The simplest choice is $F(z) = \sin z$, and one obtains

$$f(z) = \sin^2 z \sum_{k=1}^\infty A_k/(z-p\pi)^2, \qquad p = p_k,$$

under the condition $\sum_1^\infty |A_k| \cdot p_k^{-2} < \infty$, which can always be fulfilled. The question arises whether there is also a hyperbolic surface with the same property. The answer is affirmative, as shown by choosing for $F(z)$ the function $K(z)$ of Koenigs defined by $R_1(z) = z(z-s)(1-sz)^{-1}$, $R_n(z) = R_1(R_{n-1}(z))$, $K(z) = \lim_{n\to\infty} s^{-n} R_n(z)$.

L. Ahlfors (Cambridge, Mass.).

Volkovyskiĭ, L. I. The influence of the closeness of the branch points on the type of a simply connected Riemann surface. Mat. Sbornik N.S. **25(67)**, 415–450 (1949). (Russian)

As is known, a high degree of branching of an open, simply connected Riemann surface does not suffice, in general, to guarantee the hyperbolic type of the surface. Thus in Gross' example [Math. Ann. **79**, 201–208 (1918)] an entire function is constructed for which every complex value is an asymptotic value. The author studies the influence of the closeness of the branch points on the parabolic type of the surface. The principal tools are Ahlfors' sufficient condition for parabolic type and Z. Kobayashi's method [Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **2**, 125–165 (1935)]. An open, simply connected Riemann surface F lying over the w-plane with infinitely many algebraic branch points will be called a surface with algebraic points clustering about ∞ (or of class A_∞) if there exists in F a sequence of simply connected Riemann surfaces F_n $(n=1, 2, \cdots)$, each of which has not more than a finite number of algebraic branch points only and is bounded by a single closed contour lying over the circle $|w| = \rho_n$, such that (1) $\rho_1 < \cdots < \rho_n < \cdots$, $\rho_n \to \infty$; (2) $F_1 \subset \cdots \subset F_n \subset \cdots$; (3) $\lim_{n\to\infty} F_n = F$. For such a surface F, an arbitrary point P_0 over $w=0$ is fixed. If the element of F with center at P_0 is continued indefinitely over the circle $|w| < \rho$, $\rho > 0$, a subregion G_ρ of F is obtained which lies over this circle, contains P_0, and is bounded by closed contours lying over the circle $|w| = \rho$. A typical theorem proved is the following. Let V_n $(n=1, 2, \cdots)$ be an arbitrary preassigned monotonically increasing sequence of integers and let ρ_n $(n=1, 2, \cdots)$ be an arbitrary sequence of positive numbers increasing so rapidly that the series $\sum_{n=1}^\infty V_{n+1}^{-1} \log(\rho_{n+1}/\rho_n)$ diverges. For the surface F to be of parabolic type it is sufficient that the sum $V(\rho)$ of the orders of the algebraic branch points in G_ρ shall satisfy the condition $V(\rho_n) \leqq V_n$. An analogous definition and theorem are given for surfaces with algebraic branch points clustering about 0 and ∞. Furthermore, the condition that the branch points are algebraic is dropped. A similar theorem is established in the case that the arbitrary branch points cluster about ∞. Other classes of surfaces are also investigated. The paper concludes with a discussion of surfaces possessing the Gross property, particularly with a view to their parabolic type.

W. Seidel (Rochester, N. Y.).

* **Wirtinger, Wilhelm. Ueber gewisse mehrdeutige Umkehrprobleme bei Abel'schen Integralen, insbesondere beim Geschlechte Vier.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 159–169, Rome, 1943.

Suppose that $\{dv_u\}$ is a basis for the differentials of the first kind on an algebraic Riemann surface of genus p. The author considers general inversion problems

$$\sum_{\beta=1}^m q_\beta \int^{z_\beta} dv_u \equiv w_u$$

(mod periods), $1 \leqq \alpha \leqq p$ with $m \leqq p$ and integers q_β. Using coverings of the parallelotope of periods and computations of volumina the number of solutions can be evaluated in general cases for given values w_u. Connections with the geometry of curves are discussed for $p=4$ where more detailed results on the associated ϑ-series are known.

O. F. G. Schilling (Chicago, Ill.).

Oka, Kiyoshi. Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques. Bull. Soc. Math. France **78**, 1–27 (1950).

Die vorliegende siebente Abhandlung einer 1936 begonnenen Reihe [für die sechste Abhandlung siehe Tôhoku Math. J. (1) **49**, 15–52 (1942); diese Rev. **7**, 290] von weitreichenden Beiträgen des Verfassers zur Funktionentheorie mehrerer Veränderlichen beschäftigt sich mit Fragen der Idealtheorie analytischer Funktionen. Sie ist ohne Kenntnis einer 1944 erschienenen Arbeit von H. Cartan [Ann. Sci. École Norm. Sup. (3) **61**, 149–197 (1944); diese Rev. **7**, 290] entstanden, die ähnliche Fragen zum Gegenstand hatte. Es werden die folgenden Probleme behandelt. (A) In der abgeschlossenen, beschränkten Punktmenge E des Raumes der komplexen Veränderlichen $x_1, \cdots, x_n$ seien q reguläre Funktionen $F_1((x)), \cdots, F_q((x))$ sowie eine reguläre Funktion $\phi((x))$ gegeben, derart dass in einer Nachbarschaft jedes Punktes P von E gilt $\phi = \alpha_1^{(P)} \cdot F_1 + \cdots + \alpha_q^{(P)} \cdot F_q$ mit in P regulären Funktionen $\alpha_j^{(P)}((x))$ (abgekürzt $\phi((x)) \equiv 0$ [mod (F) in P]). Gesucht sind q in E reguläre Funktionen $A_1((x)), \cdots, A_q((x))$, sodass in einer Umgebung von E gilt $\phi = A_1F_1 + \cdots + A_qF_q$. (B) In E seien q reguläre Funktionen $F_1, \cdots, F_q$ gegeben; ferner sei jedem Punkte P von E ein Polyzylinder γ um P zugeordnet und eine in γ reguläre Funktion φ, sodass für jedes Paar von Polyzylindern γ und γ'' mit nichtleerem Durchschnitt δ die zugehörigen Funktionen φ' und φ'' der Relation $\varphi'((x)) \equiv \varphi''((x))$ [mod (F)] in jedem Punkte von δ genügen möge. Gesucht wird eine in E reguläre Funktion $\phi((x))$, derart dass $\phi((x)) \equiv \varphi((x))$ [mod (F)] in einer Nachbarschaft jedes Punktes von E gilt. (C) Zwei in einem Bereich D gegebene Systeme von regulären Funktionen $(f) = \{f_1, \cdots, f_r\}$ und $(\varphi) = \{\varphi_1, \cdots, \varphi_s\}$ heissen äquivalent, wenn sie in D das gleiche Ideal erzeugen; E, γ mögen die gleiche Bedeutung wie in (B) haben. In jedem γ sei ein System von dort regulären Funktionen (f) so gegeben, dass für jeden nichtleeren Durchschnitt $\delta = \gamma' \cap \gamma''$ die zugeordneten Systeme (f') und (f'') in einer Nachbarschaft jedes Punktes von δ äquivalent sind. Gesucht ist ein System (F) in E regulärer Funktionen, das in einer Nachbarschaft jedes Punktes von E mit dem dort festgelegten System (f) äquivalent ist.

Verf. zeigt, dass die Probleme (A), (B), (C) stets lösbar sind, falls E ein abgeschlossener Polyzylinder ist. In den Beweisen wird eine von H. Cartan stammende Verallgemeinerung des sogenannten Cousinschen Heftungsprozesses [J. Math. Pures Appl. (9) **19**, 1–26 (1940); diese Rev. **1**, 312] wesentlich benutzt. Zunächst werden (B) und (C) auf (A) zurückgeführt. Das Problem (A) wird auf die Aufgabe reduziert, alle lokalen Lösungen der linearen homogenen Gleichung $B_1F_1 + \cdots + B_qF_q = 0$ mit $B_1, \cdots, B_q$ als unbekannten Funktionen in bestimmter Weise durch gewisse Grundlösungen auszudrücken. Dies gelingt in einer Reihe von Schritten, wobei u.a. der folgende Satz zu Hilfe genommen wird. Sei D ein schlichter, endlicher Bereich im Raume der $x_1, \cdots, x_n$ und C ein Kreis in der Ebene der komplexen Variablen y. Im Bereiche $D \times C$ sei eine reguläre Funktion $F((x), y)$ gegeben, derart dass für jeden Punkt $(x^{(0)})$ von D die Gleichung $F((x^{(0)}), y) = 0$ genau λ Wurzeln besitzt; λ bedeute eine von $(x^{(0)})$ unabhängige ganze Zahl. Dann gestattet jede in $D \times C$ reguläre Funktion $f((x), y)$ eine Darstellung $f((x), y) = f_0((x), y) + \varphi((x), y) \cdot F((x), y)$, wo f_0 und φ in $D \times C$ regulär sind und f_0 ein Polynom in y höchstens vom Grade $\lambda - 1$ ist (oder identisch Null, falls $\lambda = 0$); diese Darstellung ist überdies eindeutig bestimmt. (Ein ähnlicher Satz wurde auch von H. Cartan bewiesen und verwendet.)

Eine wesentliche Rolle in den Untersuchungen des Verf. spielt der Begriff des "regulären Ideals ohne bestimmten Bereich" (idéal holomorphe de domaines indéterminés). Hierunter wird eine Menge (I) von Paaren (f, δ) verstanden (δ ein Bereich und f eine in δ reguläre Funktion), die folgende Bedingungen erfüllen. (1) Wenn $(f, \delta) \varepsilon (I)$ und die Funktion α in δ' regulär ist, so gilt $(\alpha \cdot f, \delta \cap \delta') \varepsilon (I)$. (2) Wenn $(f, \delta) \varepsilon (I)$ und $(f', \delta') \varepsilon (I)$, so auch $(f + f', \delta \cap \delta') \varepsilon (I)$ ($\delta \cap \delta'$ sei jeweils nicht leer). Beispiel: Die Gesamtheit (I_1) aller Paare (B_1, δ), sodass in δ reguläre Funktionen $B_2, \cdots, B_q$ existieren, die mit B_1 zusammen eine Lösung der Gleichung $B_1F_1 + \cdots + B_qF_q = 0$ bilden. Da die Lösungen $B_1, \cdots, B_q$ aus gewissen Grundlösungen zusammengesetzt werden können, folgt, dass sich für (I_1) endliche "lokale Pseudobasen" auffinden lassen. Damit ist insbesondere ein Problem gelöst, das von H. Cartan als von zentraler Bedeutung für die weitere Entwicklung der Idealtheorie analytischer Funktionen aufgewiesen worden war (in der zitierten Arbeit als "premier problème" bezeichnet). Es bleibt die Frage offen, wann ein beliebig vorgegebenes Ideal ohne bestimmten Bereich endliche lokale Pseudobasen besitzt. *K. Stein.*

Zygmund, A. On the boundary values of functions of several complex variables. I. Fund. Math. **36**, 207–235 (1949).

Extension aux fonctions $f(z_1, \cdots, z_p)$ holomorphes dans $\Gamma_p = E[|z_k| < 1]$, $1 \leqq k \leqq p$, du théorème de Fatou sur les limites radiales, et application aux séries trigonométriques de p variables. On considère seulement les limites "non tangentielles": $(z_1, \cdots, z_p) \to (e^{i\theta_1}, \cdots, e^{i\theta_p})$, où $z_k \to e^{i\theta_k}$, $1 \leqq k \leqq p$, en suivant un chemin l_k aboutissant à $e^{i\theta_k}$ et non tangent à $|z_k| = 1$. L'auteur établit: la condition

$$(1) \quad \int_0^{2\pi} \cdots \int_0^{2\pi} \log^+ |f(r_1 e^{i\theta_1}, \cdots, r_p e^{i\theta_p})| \times [\log^+ \log^+ |f(r_1 e^{i\theta_1}, \cdots, r_p e^{i\theta_p})|]^{p-1} d\theta_1 \cdots d\theta_p < M < \infty$$

pour $r_k < 1$, est suffisante pour que la limite

$$f(z_1, \cdots, z_p) \to f(e^{i\theta_1}, \cdots, e^{i\theta_p})$$

existe non tangentiellement pour presque tout $(\theta) = (\theta_1 \cdots, \theta_p)$ dans $Q_p = E[0 \leqq \theta_k < 2\pi, 1 \leqq k \leqq p]$. Avec l'hypothèse moins restrictive

$$(2) \quad \int_0^{2\pi} \cdots \int_0^{2\pi} \log^+ |f(r_1 e^{i\theta_1}, \cdots, r_p e^{i\theta_p})| \times d\theta_1 \cdots d\theta_p < M < \infty$$

l'auteur démontre: la limite

$$f_1(e^{i\theta_1}, z_2, \cdots, z_p) = \lim f(z_1, \cdots, z_p),$$

pour $z_1 \to e^{i\theta_1}$ non tangentiellement existe, uniformément pour $|z_k| \leqq r < 1$, $k \geqq 2$, pour presque tout θ_1; elle est une fonction analytique des z_k, $k \geqq 2$, et satisfait

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \log^+ |f_1(e^{i\theta_1}, r_2 e^{i\theta_2}, \cdots, r_p e^{i\theta_p})| < M_1(\theta_1) < \infty$$

pour presque tout θ_1. De proche en proche on définit la limite itérée $f_{1 \cdots p}(\theta_1, \cdots, \theta_p)$ en passant à la limite successivement sur $z_1 \to e^{i\theta_1}, \cdots, z_p \to e^{i\theta_p}$; elle existe presque partout dans Q_p.

Le premier théorème repose sur le lemme: $\Delta(\theta, \eta)$ désignant le domain limité par les tangentes menées du point $e^{i\theta}$ au cercle $|z| = \eta < 1$, et par le grand arc du cercle $|z| = \eta$, l'hypothèse (1) entraine que pour presque tout (θ), f est bornée dans le domaine $\Delta(\theta_1, \eta) \times \cdots \times \Delta(\theta_p, \eta)$; pour établir le second résultat l'auteur montre que, avec l'hypothèse (2), f est continue des $p-1$ variables restantes z_k dans $|z_k| \leqq r < 1$, quand $z_1 \varepsilon \Delta(\theta_1^0, \eta)$, $\eta < 1$. Avec l'hypothèse (1) la limite f est

égale aux limites itérées $f_{i_1 i_2 \cdots i_p}$, alors égales entre elles. En particulier le classe H^a vérifie (1), où H^a est definie par $\int |f|^a d\theta_1 \cdots d\theta_p < M_a < \infty$. Pour que la serie

$$\sum_{n_k=-\infty}^{\infty} c_{n_1 \cdots n_p} e^{i(n_1\theta_1 + \cdots + n_p\theta_p)} = \varphi$$

soit la partie réelle d'une fonction $f(z_1, \cdots, z_p)$ pour $z_k = e^{i\theta_k}$, il faut et il suffit que les coefficients ayant deux indices de signes différents soient nuls; on a alors

$$\varphi = \sum_{n_k=0}^{\infty} A_{n_1 \cdots n_p} \cos(\sum n_k\theta_k) + B_{n_1 \cdots n_p} \sin(\sum n_k\theta_k).$$

L'auteur appelle conjuguée de φ la serie formelle

$$\bar{\varphi} = \sum_{n_k=0}^{\infty} A_{n_1 \cdots n_p} \sin(\sum n_k\theta_k) - B_{n_1 \cdots n_p} \cos(\sum n_k\theta_k)$$

et établit: si $\varphi(x_1, \cdots, x_p)$ est périodique (2π) de chaque x_i, si $|\varphi|(\log^+|\varphi|)^{p-1} \varepsilon L^1$, alors, si $Q = E[|x_k| \leqq \pi]$ et $a = E[|x_k| < \epsilon_k]$, la limite

$$\bar{\varphi}(x_1, \cdots, x_p) = \lim_{\epsilon_k \to 0} (-\pi)^p \int_{Q-a} \varphi(x_1+u_1, \cdots, x_p+u_p) \times \tfrac{1}{2}\cot(\tfrac{1}{2}u_1) \cdots \tfrac{1}{2}\cot(\tfrac{1}{2}u_p) du_1 \cdots du_p$$

existe presque partout et

$$\int_Q (|\varphi|^\delta dx_1 \cdots dx_p)^{1/\delta} \leqq A_{\delta,p} \int_Q |\varphi|(\log^+|\varphi|)^{p-1} dx_1 \cdots dx_p + A_{\delta,p}.$$

P. Lelong (Paris).

* **Calderón, A. P., and Zygmund, A. Note on the boundary values of functions of several complex variables.** Contributions to Fourier Analysis, pp. 145–165. Annals of Mathematics Studies, no. 25. Princeton University Press, Princeton, N. J., 1950. $3.00.

Les auteurs complètent un travail précédent [voir l'analyse ci-dessus] et montrent que si $f(z_1, \cdots, z_p)$ est analytique dans $D = E[|z_k| < 1, 1 \leqq k \leqq p]$ et satisfait à

$$\int \log^+ |f(r_1 e^{i\theta_1}, \cdots, r_p e^{i\theta_p})| d\theta_1 \cdots d\theta_p < M < \infty,$$

alors f possède en presque toute point de l'arête

$$\Gamma = E[z_k = e^{i\theta_k},\ 0 \leqq \theta_k \leqq 2\pi,\ 1 \leqq k \leqq p]$$

une limite finie quand $z_k = r_k e^{i\varphi_k} \to e^{i\theta_k}$ sur Γ avec les restrictions suivantes: (a) chaque z_k tend vers $e^{i\theta_k}$ non tangentiellement au cercle $|z_k| = 1$; (b) tous les rapports $(1-r_j)/(1-r_k)$ demeurent bornés. La démonstration utilise la propriété analogue de l'intégrale de Poisson-Stieltjes de p variables établie antérieurement [Zygmund, Amer. J. Math. 69, 836–850 (1947); ces Rev. 9, 235]. *P. Lelong* (Lille).

Peschl, E., und Müller, Cl. Zur Verallgemeinerung des Schwarzschen Lemmas auf mehrere Dimensionen. Proc. Cambridge Philos. Soc. 46, 396–405 (1950).

The authors introduce a generalization $D_n(B; P; Q)$ of Carathéodory's metric as follows. Let B be a domain in the space of two complex variables x, y and let P be a point of B. Denote by $F(B; n, P)$ the family of regular functions $f(x, y)$ in B for which $|f| \leqq 1$ in B and $\partial^{\mu_1+\mu_2} f / \partial x^{\mu_1} \partial y^{\mu_2} = 0$ at P for all $\mu_1 + \mu_2 \leqq n$. Then for $Q \varepsilon B$ they set

$$D_n(B; P; Q) = \max_{f \varepsilon F} |f(Q)|.$$

Next let K be a (circular) domain which with the point (x, y) contains also all points $(\rho x, \rho y)$ with $|\rho| \leqq 1$. Assume that K is normalized so that it lies in the bicylinder $|x| \leqq 1$, $|y| \leqq 1$ and so that the circle $|x| = 1$, $y = 0$ belongs to the boundary of K. Finally assume that K is convex with respect to all homogeneous polynomials $A_n(x, y)$. This latter requirement is equivalent to asking that K be the region of uniform convergence of a series of homogeneous polynomials. Denote by $h_n(K)$ the point set for which $|A_n(x, y)| \leqq 1$ holds for all homogeneous polynomials $A_n(x, y)$ of degree n which satisfy the inequality $|A_n| \leqq 1$ in K. Finally, for positive ρ, denote by $K_n(\rho)$ the point set defined by $D_n(K; 0; Q) \leqq \rho$ and for positive p denote by $pK_n(\rho)$ the set obtained from $K_n(\rho)$ by means of the similarity transformation $x' = px$, $y' = py$. The first main theorem states that

$$\lim_{\rho \to \infty} \rho^{-1} K_n(\rho^n) = h_n(K)$$

with the customary definition of limits of point-sets. The second main theorem states that if $x' = \varphi_1(x, y)$, $y' = \varphi_2(x, y)$ is an inner mapping of K, and if we develop φ_1 and φ_2 in series of homogeneous polynomials

$$\varphi_j = Q_{j,N}(x, y) + Q_{j,N+1}(x, y) + \cdots \qquad (j = 1, 2),$$

where $Q_{j,\mu}$ is a homogeneous polynomial of degree μ, then the mapping $x' = Q_{1,N}(x, y)$, $y' = Q_{2,N}(x, y)$ defined with the beginning homogeneous polynomials is also an inner mapping of K. They first prove these results for the special case of circular domains $K(M)$ which are normalized as above but which are assumed convex only with respect to all homogeneous polynomials of degree $\mu \leqq M$ for an arbitrary but fixed positive integer M. They obtain the general theorem stated above by a simple limiting argument on M.

W. T. Martin (Cambridge, Mass.).

Slivinskiĭ, V. E. Concerning analytic functions of the variables of Zolotarev-Krylov. Doklady Akad. Nauk SSSR (N.S.) 72, 465–468 (1950). (Russian)

Krylov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 683–684 (1947); these Rev. 9, 233] has studied analytic functions of a variable of the form $\sum_{k=0}^{n-1} \alpha_k i^k$, where i is a root of an algebraic equation of degree n. Extending this investigation, the author considers a square matrix X with given characteristic equation $\omega^n = \sum_{k=0}^{n-1} p_k \omega^k$, and for the independent variable $Z = \sum_{k=0}^{n-1} x_k X^k$, where the x_k are real, he studies functions of the form $W = \sum_{k=0}^{n-1} u_k X^k$, where the u_k are smooth functions of $x_0, \cdots, x_{n-1}$. It is pointed out that in this system the law of commutation for multiplication holds, and that division except by "zero" is always possible and unique. Conditions for analyticity, corresponding to the Cauchy-Riemann equations, are obtained.

E. F. Beckenbach (Los Angeles, Calif.).

Fedorov, V. S. The monogenic vector function. Doklady Akad. Nauk SSSR (N.S.) 72, 637–639 (1950). (Russian)

A monogenic vector function is a generalization of a monogenic function of a complex variable. If $\mathbf{r}$ is the radius vector of a variable point of a region D in three-dimensional space, and if $\mathbf{f}(\mathbf{r})$ is a vector function of $\mathbf{r}$, then $\mathbf{f}(\mathbf{r})$ is said to be monogenic at a point M of D if $\mathbf{f}(\mathbf{r})$ possesses a derivative $\partial\mathbf{f}/\partial s$ at M which does not depend on the direction along which the increment $\Delta\mathbf{r}$ approaches zero. It is proved that if $\mathbf{f}(\mathbf{r})$ is monogenic at every point of D, then, in D, $\mathbf{f}(\mathbf{r})$ is a linear function of the radius vector $\mathbf{r}$.

A. J. Lohwater (Ann Arbor, Mich.).

Theory of Series

de Castro Brzezicki, A. On continuable and noncontinuable Taylor series. Gaceta Mat. (1) 1, 263–268 (1949). (Spanish)

Expository article. *R. C. Buck* (Madison, Wis.).

Barrucand, René. Sur les suites réciproques. C. R. Acad. Sci. Paris 230, 1727–1728 (1950).

Two sequences, $\{f(n)\}$ and $\{g(n)\}$ are called reciprocal if, for $n=0, 1, 2, \cdots$, $f(n)=\sum_{m=0}^{n}(-)^m\binom{n}{m}g(m)$ and $g(n)=\sum_{m=0}^{n}(-)^m\binom{n}{m}f(m)$. The author points out that $\{t^n/n!\}$ and $\{L_n(t)\}$ are reciprocal, deduces the expansion

$$e^t-\int_0^\infty[t^s/\Gamma(1+s)]ds=\sum i_{n+1}L_n(t),$$

where i_n is the coefficient of x^n in the expansion of $x/\log(1-x)$, and obtains various relationships connecting certain integrals, some of them related to Ramanujan's work. He concludes with the remark that if two functions are connected by the Hankel transformation of order zero, with kernel $J_0(2(xy)^{\frac{1}{2}})$, then the coefficients of their expansions in Laguerre polynomials form reciprocal sequences.

A. Erdélyi (Pasadena, Calif.).

Levin, V. I. Concerning a problem of S. Ramanujan. Uspehi Matem. Nauk (N.S.) 5, no. 3(37), 161–166 (1950). (Russian)

Let $\zeta_k(x)=\sum_{n=1}^{\infty}n^{-k}x^n$. Then

$$\zeta_1(\tfrac{1}{2})=\log 2,\quad \zeta_2(\tfrac{1}{2})=-\tfrac{1}{2}\log^2 2+\pi^2/12,$$

and Ramanujan [Collected Papers, Cambridge University Press, 1927, pp. xxv, 337] gave the formula

$$\zeta_3(\tfrac{1}{2})=\tfrac{1}{6}\log^3 2-\tfrac{1}{12}\pi^2\log 2+\sum_{m=1}^{\infty}(2m-1)^{-3}.$$

The author establishes, for $k=1, 2, \cdots$, that

$$\zeta_k(\tfrac{1}{2})=\frac{(-1)^{k-1}}{k!}\log^k 2+\sum_{s=0}^{k}\frac{(-1)^{k-s}}{(k-s)!}C_s\log^{k-s}2,$$

where

$$C_s=\zeta_s(1)-\frac{1}{(s-1)!}\int_0^1 y^{-1}\log^{s-1}(1+y)dy.$$

He shows that this reduces to Ramanujan's formula for $k=3$.

R. P. Boas, Jr. (Evanston, Ill.).

Cherry, T. M. Summation of slowly convergent series. Proc. Cambridge Philos. Soc. 46, 436–449 (1950).

To compute the sum $s=s_n+R_n$ of a slowly convergent power series of general term C_rt^r whose partial sum s_n of the first n terms is obtained by direct addition of terms, the author transforms the remainder R_n into a divergent series which represents R_n asymptotically and evaluates the upper bound of the error made by considering the sum of the first p terms of this divergent asymptotic series as an approximate value of R_n. Two such transformations are defined and studied (formulae 6 and 7). They are based on a decomposition of the numerical coefficient C_r into a product $c_r\cdot f(r)$ of two factors c_r and $f(r)$. The choice of the function $f(z)$ and thus of the numerical factor $c_r=C_r/f(r)$ is arbitrary, provided the following conditions are fulfilled: (a) the analytic function $f(z)$ is regular in the half-plane $\Re(z)>0$, (b) for $z\to\infty$ in the same half-plane, $f(z)$ has an asymptotic expansion $f(z)\sim A_1z^{k_1}+A_2z^{k_2}+\cdots$, for which the exponent differences k_i-k_{i+1} have a positive lower bound, (c) the power series $F_n(t)=\sum_{m=0}^{\infty}c_{n+m}t^m$ has a finite radius of convergence, and (d) explicit closed expressions for the function $F_n(t)$ and its derivatives must be deducible from their power series to enable the evaluation of $F_n(t)$ and its derivatives (since their power series converge too slowly). The conditions (a)–(d) mean practical restrictions upon the choice of $f(z)$, that is, restrictions on the slowly convergent series whose sums could be computed effectively with the aid of the two transformations considered by the author.

E. Kogbetliantz (New York, N. Y.).

Vernotte, Pierre. L'interpolation idéale par les expressions non uniformes. C. R. Acad. Sci. Paris 230, 2000–2002 (1950).

Remarks concerning difficulties met in applying the author's ideas of "interpolation idéale" [Théorie et pratique des séries divergentes . . . , Publ. Sci. Tech. Ministère de l'Air, Paris, no. 207 (1947); these Rev. 11, 97] to simple periodic sequences. *R. P. Agnew* (Ithaca, N. Y.).

Vermes, P. Certain classes of series to series transformation matrices. Amer. J. Math. 72, 615–620 (1950).

Let A be the infinite square matrix of a series-to-series transformation, and F_A the matrix of the corresponding sequence-to-sequence transformation. Let I be the identity matrix (with elements $i_{nk}=\delta_{nk}$), E^h the matrix obtained by adjoining h rows of zeros to the upper edge of I ($e_{nk}^{(h)}=\delta_{n+h,k}$), E' the transpose of $E^1=E$, R the matrix $\sum_{h=0}^{\infty}E^h$, and S the matrix $I-E$. The author shows that $F_A=RAS$, and he solves the equation (1) $RXS=pX$, where the symbol p represents an arbitrary complex number; that is, he determines the class of Toeplitz transformations whose matrices are scalar multiples of the matrices of the corresponding series-to-series transformations. The solution of the equation (1) is $X=DA(p)$; here D is any matrix of the form $\sum_{n=0}^{\infty}d_nE^n$, where $\{d_n\}$ is an arbitrary sequence of numbers; and $A(p)$ is the upper-triangular matrix with elements $a_{nk}=\binom{k}{n}p^n(1-p)^{k-n}$ for $k\geqq n$. The matrix $A(p)$ is intimately associated with the analytic continuation of Taylor series [for a bibliography on $A(p)$, see Agnew's review of a paper by Meyer-König [these Rev. 11, 242]]. The author also solves the equations $RXS=pXE$, $RXS=pE'X$, and $RXS=pE'XE$.

G. Piranian (Ann Arbor, Mich.).

Hill, J. D. Summability methods weaker than convergence. Amer. J. Math. 72, 621–623 (1950).

Constants C_n, $n=1, 2, 3, \cdots$, and $C_{n,k}$, $n, k=1, 2, 3, \cdots$, for which the C_n are not all zero define a W-method by which a sequence s_k is evaluable to L if

$$\lim_{n\to\infty}[C_n\lim_{k\to\infty}s_k+\sum_{k=1}^{\infty}C_{nk}s_k]=L.$$

Let G be a linear proper subset of the Banach space c of convergent sequences. In order that there exist a W-method which evaluates those and only those sequences in G, it is sufficient (but not necessary) that G be closed.

R. P. Agnew (Ithaca, N. Y.).

Barrucand, Pierre. Transformation de Laplace et formules sommatoires. C. R. Acad. Sci. Paris 231, 20–22 (1950).

Let $[x]$ be the largest integer not exceeding x and let $A(x)=\sum_0^{[x]}a(n)$, the prime indicating that if x is an integer $a(x)$ is taken with a factor $\frac{1}{2}$. We have

$$\int_0^\infty e^{-tx}A(x)dx=t^{-1}\sum_0^\infty a(n)e^{-nt}.$$

If there exists a second development

$$\sum a(n)e^{-nt}=\sum b(n)\psi(t,n),$$

and if there exists $H_1(x,t)$ such that

$$\int_0^\infty e^{-xt}H_1(x,n)dx=t^{-1}\psi(t,n),$$

then formally $A(x)=\sum b(n)H_1(x,n)$ and consequently

$$\sum a(n)f(n)=\int_0^\infty f(x)dA(x)=\sum b(n)\int_0^\infty f(x)dH_1(x,n).$$

Special examples are the summation formulas of Poisson and of Landau-Voronoï. *I. I. Hirschman.*

Szász, Otto. On a summation method of O. Perron. Math. Z. **52**, 631–636 (1950).

The author discusses two closely connected methods of summability. The first of these, due to Perron [Math. Z. **6**, 286–310 (1920)] and called ϕ_λ-summability, is defined by

$$\lim_{x\to\infty}\left\{a_0+\sum_{\nu=1}^{\infty}a_\nu\frac{x}{x+\lambda}\frac{x+1}{x+\lambda+1}\cdots\frac{x+\nu-1}{x+\lambda+\nu-1}\right\},\qquad\lambda>0.$$

The second method, ϕ_λ^*, is defined by

$$\lim_{x\to\infty}\left\{\frac{\Gamma(x+\lambda)}{\Gamma(x)\Gamma(\lambda)}\int_0^\infty\frac{\rho^{x-1}}{(1+\rho)^{x+\lambda}}\left(\sum_{\nu=0}^{\infty}a_\nu\left(\frac{\rho}{1+\rho}\right)^\nu\right)d\rho\right\},\qquad\lambda>0.$$

It is shown that (i) if $\sum a_n$ is Abel summable, then it is ϕ_λ^*-summable for any $\lambda>0$ to the same value, (ii) if $\sum a_n$ is (C,λ)-summable, then it is ϕ_λ-summable, and (iii) the ϕ_λ transform of a Fourier series does not present a Gibbs phenomenon. *H. G. Eggleston* (Swansea).

Rajagopal, C. T. On an absolute constant in the theory of Tauberian series: Postscript. Proc. Indian Acad. Sci., Sect. A. **31**, 60–61 (1950).

Methods and results of the author [same Proc., Sect. A. **28**, 537–544 (1948); these Rev. **10**, 447] and the reviewer [Ann. of Math. (2) **50**, 110–117 (1949); these Rev. **10**, 291] are combined to complete the author's theorem about transforms of the type $F(s)=\sum a_n\exp(-\lambda_n s)$. *R. P. Agnew* (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

***Wang, Shou-jen. On the absolute convergence of the Fourier series of a function of bounded variation.** National Peking University Semi-Centennial Volume, Mathematical, Physical and Biological Series, pp. 26–30, 1948.

It is well known that if a function $f(x)$ of period 2π and with Fourier coefficients a_n, b_n is both of bounded variation and of the class Lip α, $0<\alpha\leqq 1$, then both series

$$(*)\qquad\sum(|a_n|^\beta+|b_n|^\beta),\quad\sum n^{\beta\gamma}(|a_n|+|b_n|)$$

converge for $\beta>2/(2+\alpha)$, $\gamma<\frac{1}{2}\alpha$; the series (*) can diverge if $\beta=2/(2+\alpha)$ or $\gamma=\frac{1}{2}\alpha$ [see Zygmund, Trigonometrical Series, Warszawa-Lwów, 1935, chapter 6]. The author shows that the series (*) do converge in the latter cases if the condition $f\in$Lip α is replaced, respectively, by the conditions

$$\omega(t)=O\{t^\alpha[l_1(t)l_2(t)\cdots l_k^{1+\epsilon}(t)]^{-2-\alpha}\},$$
$$\omega(t)=O\{t^\alpha[l_1(t)l_2(t)\cdots l_k^{1+\epsilon}(t)]^{-2}\},$$

where $\omega(t)$ is the modulus of continuity of f, ϵ is positive, and $l_1(t)=\log(1/t)$, $l_2(t)=\log\log(1/t)$, $\cdots$ for t sufficiently small. *A. Zygmund* (Cambridge, Mass.).

Ditkin, V. A. On the completeness of a system of trigonometric functions. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 196–197 (1950). (Russian)

The author proves that the system $\{\cos nx+\lambda_n\sin nx\}$, $n=0,1,2,\cdots$, is complete (that is, is orthogonal to no integrable function which fails to vanish almost everywhere) on $(0,\pi)$ provided all $\lambda_n\geqq 0$. *R. P. Boas, Jr.*

Leont'ev, A. F. On a sequence of polynomials. Doklady Akad. Nauk SSSR (N.S.) **72**, 621–624 (1950). (Russian)

The author shows that if $\lambda_n\uparrow\infty$ and $\sum 1/\lambda_n<\infty$, then the uniform convergence of $\{\sum_{\nu=1}^{n}a_\nu x^{\lambda_\nu}\}$ on $0\leqq x\leqq 1$ entails uniform convergence in $0<|x|<1$, $|\arg x|<\infty$, and so the analyticity of the limit function. He was evidently unaware that this had been proved before by Clarkson and Erdös [Duke Math. J. **10**, 5–11 (1943); these Rev. **4**, 196] for integral λ_n and independently by L. Schwartz [Étude des sommes d'exponentielles réelles, Actualités Sci. Ind., no. 959, Hermann, Paris, 1943; these Rev. **7**, 294] in the general case. For further results see Hirschman [Ann. of Math. (2) **50**, 666–675 (1949); these Rev. **10**, 702]. *R. P. Boas, Jr.* (Evanston, Ill.).

Bohman, H. On a class of orthogonal series. Ark. Mat. **1**, 13–19 (1949).

The author generalizes a theorem of Kolmogoroff which states that if $\{\varphi_n\}$ is an orthonormal system of independent random variables on $(0,1)$, then $\sum a_n^2<\infty$ is a necessary and sufficient condition for the convergence almost everywhere of the series $\sum a_n\varphi_n(x)$. The system $\{\varphi_n\}$ is interpreted as a system of real functions on the torus space Q_ω [Jessen, Acta Math. **63**, 249–323 (1934)] consisting of vectors $\xi=(x_1,\cdots,x_n,\cdots)$, $0\leqq x_n<1$, with the property that $\varphi_n(\xi)=\varphi_n(x_n)$; that is, for each n, $\varphi_n(\xi)$ depends only on x_n. If $\varphi_n(x_n)$ is an orthonormal system of this type, a new orthonormal system $\{\psi_n(\xi)\}$ is defined by $\psi_n=\varphi_{\nu_1}\varphi_{\nu_2}\cdots\varphi_{\nu_k}$ if $n=2^{\nu_1}+2^{\nu_2}+\cdots+2^{\nu_k}$ is the expression of n in the dyadic scale. [This generalizes Paley's construction of the Walsh functions from the Rademacher functions.] If the distribution functions $F_n(t)=mE[\varphi_n(x)\leqq t]$ are all continuous, then the series $\sum a_n\psi_n$ is either convergent almost everywhere or divergent almost everywhere. Denote a function ψ by $\psi^{(k)}$ if it has k factors φ_{ν_i}. The system $\{\psi_n\}$ is thus divided into partial systems $\{\psi_n^{(k)}\}$, $k=1,2,\cdots$; in each partial system the lower indices are changed, while the mutual order is kept unaltered. Then if $\sum a_n^2<\infty$, the series $\sum a_n\psi_n^{(k)}$ is convergent almost everywhere for every k. *N. J. Fine.*

Herrera, Félix Eduardo. On the equiconvergence of Fourier series and Fourier integrals. Bull. Amer. Math. Soc. **55**, 1182–1190 (1949).

The author states that he considers the convergence and (C,k)-summability of $\int_0^\infty(a(u)\cos ux+b(u)\sin ux)du$ which is such that

$$a(u)=\pi^{-1}\int_{-\infty}^{\infty}f(t)\cos utdt,\quad b(u)=\pi^{-1}\int_{-\infty}^{\infty}f(t)\sin utdt$$

and (*) $\int_{-\infty}^{\infty}\{|f(t)|/(1+|t|)\}dt<\infty$. It is not stated in what sense the integrals defining $a(u)$ and $b(u)$ exist, but the author presupposes that for

$$S_\omega=\int_0^\omega(a(u)\cos ux+b(u)\sin ux)du$$

the expression $\pi^{-1}\int_{-\infty}^{\infty}f(x+t)(\sin\omega t/t)dt$ can be substituted, so that it would be more accurate to say that the convergence and the summability for $\omega\to\infty$ of the last function is being

investigated. The following theorems are proved. Given any function satisfying (*) and any interval $I_a=(a, a+2\pi)$ of length 2π, let $f_a(x)$ be the function of period 2π and coinciding with $f(x)$ in I_a, and let s_n denote the partial sums of the Fourier series of $f_a(x)$. Then the difference S_n-s_n tends to zero at every point inside I_a and the convergence is uniform in every closed interval interior to I_a. Another theorem deals with the case of the corresponding allied trigonometric expansions. Two more theorems deal with the k times differentiated expansions, summed (C, k). Although the author does not mention it, there exist general theorems proved by the reviewer which cover the author's problems as a special case [Univ. California Publ. Math. (N.S.) 1, 159–227 (1947); these Rev. 9, 140]. Because of condition (*), $f(t)$ can be represented by a "Stieltjes integral" $\int_0^\infty(\cos uxdA(u)+\sin uxdB(u))$ where $A(u), B(u)=o(u)$ for large u. The corollary in § 38 of the reviewer's paper yields the author's two theorems. The remaining results can be obtained in a similar way. *František Wolf.*

Sunyer i Balaguer, Ferran. A new generalization of almost periodic functions. Institut d'Estudis Catalans. Arxius de la Secció de Ciències no. 17, 46 pp. (1949). (Catalan)

Two abstracts of this paper have previously been published [C. R. Acad. Sci. Paris 228, 732–734, 797–799 (1949); these Rev. 10, 525]. It is noted that Bessonof's notes on the topic [ibid. 182, 1011–1013 (1926); 186, 63–65 (1928)] contain many errors, some of which were corrected by Norgil [Mat. Tidsskr. B. 1930, 73–91]. *E. Følner.*

Tornehave, Hans. A theorem on the mean motions of almost periodic functions. Danske Vid. Selsk. Mat.-Fys. Medd. 25, no. 20, 18 pp. (1950).

If $f(t)$ is a complex uniformly almost periodic function and is bounded away from zero for $-\infty<t<\infty$, then it is known that $\arg[f(t)]-ct$ is uniformly almost periodic for some real constant c, called the mean motion of $f(t)$. The author proves that if $F(z, w)$ is an entire function of the complex variables z, w and f, g are uniformly almost periodic functions bounded away from zero with mean motions c, d, then the relationship $F[f(t), g(t)]=0$, $-\infty<t<\infty$, implies that $mc=nd$ for some nontrivial pair of integers m, n.

R. H. Cameron (Minneapolis, Minn.).

Levin, B. On quasi-analytic classes of almost periodic functions. Doklady Akad. Nauk SSSR (N.S.) 70, 949–952 (1950). (Russian)

Soit $f(t)$ une fonction périodique (de période 2π), mesurable, et soit

$$\theta(\alpha)=\int_{t_0}^{t_0+\alpha}|f(t)|\,dt. \tag{1}$$

Soit $f(t)\sim\sum(a_{n_i}\cos n_it+b_{n_i}\sin n_it)$, et désignons par $N(r)$ le nombre des n_i inférieurs à r $(r>0)$. En posant

$$\rho=\limsup_{\alpha\to+0}[\log(\log\theta(\alpha)/\log\alpha)], \tag{2}$$

$$\sigma=\limsup_{r\to\infty}\log N(r)/r, \tag{3}$$

Mandelbrojt a démontré le théorème suivant [Séries de Fourier et classes quasi-analytiques de fonctions, Gauthier-Villars, Paris, 1935]: si $\sigma<1$, $\rho>\sigma/(1-\sigma)$, alors $f(t)\equiv0$ (presque partout). L'auteur et Lifschetz ont généralisé ce théorème tout en fournissant une démonstration plus simple [Rec. Math. [Mat. Sbornik] N.S. 9(51), 693–711 (1941); ces Rev. 3, 106]. Dans le présent travail le théorème est généralisé pour les fonctions presque-périodiques. Soit $f(t)$ une fonction presque-périodique, $f(t)\sim\sum a_k e^{i\lambda_k t}$. Désignons par $N(r)$ le nombre de μ_k compris dans $(-r, r)$ et par $A(\alpha)$ une fonction croissante de α telle que $\theta(\alpha)\leqq A(\alpha)$ (θ défini par (1)). Soit $\psi(r)$ la fonction inverse de la fonction $r=-\log A(\alpha)/\alpha$, et soit $\zeta(r)=2r^2\int_0^\infty[N(t)/t(t^2+r^2)]dt$. Le théorème s'énonce comme suit. Si

$$\liminf[3\zeta(2r)-r\psi(r)]=-\infty, \tag{4}$$

alors $f(t)\equiv0$ (presque partout). En posant

$$\varphi(z)=\int_0^\infty e^{-zt}f(t)dt,\quad M=\sup|f(t)|\qquad(t\geqq t_0=0),$$

on trouve pour tout $\alpha>0$, $x=\Re z$: $|\varphi(z)|\leqq A(\alpha)+Mx^{-1}e^{-\alpha x}$. En particulier si $\alpha=\psi(x)$: $|\varphi(z)|\leqq(1+Mx^{-1})e^{-x\psi(x)}$. En posant $\chi(z)=\varphi(z)\prod(1+z^2)/\mu_n^2$, on voit que

$$|\chi(z)|\leqq c\exp[\zeta(r)-x\psi(x)]\quad(x>0),$$
$$|\chi(z)|<c\exp\zeta(r)\quad(x<0)\quad(|z|=r).$$

Posons $\varphi(z)=\chi(z)\chi(\omega z)\chi(\omega^2 z)$ $(\omega=\exp(2\pi i/3))$. On obtient $|\varphi(z)|<c\exp[3\zeta(2x)-x\psi(x)]$ pour $x>0$, $y\leqq3^{\frac12}x$ $(y=\Im z)$. De $\varphi(\omega z)=\varphi(z)$ il résulte que la dernière inégalité a lieu sur la frontière d'un triangle équilatéral de centre à l'origine et d'apothème égal à $|x|$. De la condition (4) il résulte que $\varphi(z)\equiv0$. *S. Mandelbrojt* (Houston, Tex.).

Kahane, Jean-Pierre, et Lalaguë, Pierre. Quasi-analyticité des fonctions sommes de séries de Fourier lacunaires. C. R. Acad. Sci. Paris 230, 2250–2252 (1950).

Let (*) $f(x)=a_0+\sum_1^\infty(a_j\cos\lambda_jx+b_j\sin\lambda_jx)\not\equiv0$, where $\{\lambda_j\}$ is an increasing sequence of real numbers and $\sum(|a_j|+|b_j|)<\infty$. The authors first establish a relation between the gaps of the series and the "smallness" of the function near a fixed point. This problem, in a less general form, was treated by Mandelbrojt [Séries de Fourier . . . , Gauthier-Villars, Paris, 1935]. The authors use their result to obtain a theorem which contains previous results of Mandelbrojt [loc. cit.], Lévine and Lifschetz [Rec. Math. [Mat. Sbornik] N.S. 9 (51), 693–711 (1941); these Rev. 3, 106], and Levin [see the preceding review]. A somewhat less general statement of their theorem is the following. Let (*) be satisfied with $f(x)\in C^\infty$, $f^{(n)}(0)=0$, $|f^{(n)}(x)|<m_n$, $n\geqq0$; if, moreover, $j^{1+\epsilon}\lambda_j^{-2}$ is decreasing for some $\epsilon>0$, then $\sum(j\lambda_j)^{-1}(m_{2j+1}/m_{2j-1})^{\frac12}=\infty$. *S. Agmon.*

Barrucand, Pierre. Sur les transformées de Fourier et de Mellin des inverses de fonctions de Bessel. C. R. Acad. Sci. Paris 231, 102–104 (1950).

Let κ_n be the nth positive zero of $J_0(x)$, and $\kappa_n'=1/J_1(\kappa_n)$; put $Z(s)=\sum_{n=1}^\infty\kappa_n'\kappa_n^{-s}$. The author establishes the formula

$$\int_0^\infty\frac{x^{s-1}}{2I_0(x)}dx=\tfrac12\pi Z(1-s)\csc(\tfrac12\pi s),$$

which gives the analytic continuation of $Z(s)$ into the left half plane. *R. P. Boas, Jr.* (Evanston, Ill.).

Gross, B. Note on the inversion of the Laplace transform. Philos. Mag. (7) 41, 543–544 (1950).

The author inverts the Laplace transform by taking its Laplace transform and applying a known inversion formula to the resulting Stieltjes transform. [Cf. Widder, The Laplace Transform, Princeton University Press, 1941, pp. 384 ff.; these Rev. 3, 232.] *R. P. Boas, Jr.*

Vodička, Václav. La base théorique de la transformation de Laplace avec une application dans la physique mathématique. Časopis Pěst. Mat. Fys. **75**, D119–D131 (1950). (Czech. French summary)

This expository paper is based on two lectures given by the author. In the first lecture it is shown that the Laplace transform theory has its origin in Fourier integrals in the complex domain, and in the second lecture the author describes the application of the Laplace transformation to the solution of partial differential equations. *A. Erdélyi.*

Poli, Louis. Règles pour le calcul symbolique. Nouveau théorème du produit. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 155–164 (1949).

Let $f(p)$ denote the image of a function $h(t)$ in Heaviside operational calculus. The author begins with a known formula that gives the original function corresponding to $f(p^\lambda)$ as an integral containing $h(t)$. He transforms the integral so that it represents the image of $h(t^\lambda)/t^\lambda$. The use of this algebraic relation between the original of $f(p^\lambda)$ and the image of $h(t^\lambda)/t^\lambda$ is illustrated. In particular, the relation furnishes new rules for getting the image of $h(t^2)t$ and the original corresponding to $f(p^2)$. From a second known formula he observes that the image of $h(t^2)$ has a simple algebraic relation to the image of $f(1/t)/t^\lambda$, and this relation furnishes rules for getting the images of $h(t^\lambda)$ or $h(t^2)$ and the originals for $p^\lambda f(p^\lambda)$ and $f(p^2)/p$. Another simple derivation indicates that the image of $\int_0^\infty h_1(x)f_2(x)x^{-1}dx$ is $\int_p^\infty h_2(x-p)f_1(x)x^{-1}dx$. The use of this product rule is illustrated. The derivations here are formal, even though several of these rules apply to quite limited classes of functions, so that conditions of validity would be especially desirable.

R. V. Churchill (Ann Arbor, Mich.).

Hartman, Philip, and Wintner, Aurel. On Euler transforms. Proc. Amer. Math. Soc. **1**, 394–396 (1950).

Let E_λ denote the class of functions $\phi_\lambda(x)$, $0<x<\infty$, representable in the form

$$\phi_\lambda(x)=\int_0^\infty (x+t)^{-\lambda}d\phi(t) \qquad (\lambda>0),$$

where $\phi(t)$ is nondecreasing. It is demonstrated that if $\lambda<\mu$ then $E_\lambda \subset E_\mu$, the inclusion being proper. Thus $E_\infty=\lim_{\lambda\to\infty} E_\lambda$ is defined. Let E^∞ be the class of functions $\phi^\infty(x)$, $0<x<\infty$, representable in the form $\phi^\infty(x)=\int_0^\infty e^{-xt}d\phi(t)$, where $\phi(t)$ is nondecreasing. It is shown that $E_\infty \subset E^\infty$ the inclusion being proper, and that in a certain natural topology, E_∞ is dense in E^∞. *I. I. Hirschman* (St. Louis, Mo.).

Schoenberg, I. J. On Pólya frequency functions. II. Variation-diminishing integral operators of the convolution type. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 97–106 (1950).

[For a résumé of part I see Proc. Nat. Acad. Sci. U. S. A. **33**, 11–17 (1947); these Rev. **8**, 319.] This paper contains proofs of extensions of results announced in a previous note [Proc. Nat. Acad. Sci. U. S. A. **34**, 164–169 (1948); these Rev. **9**, 415]. The main difference is that the operator is now a Stieltjes convolution $g(x)=\int_{-\infty}^{\infty} f(x-t)dL(t)$, where $L(t)$ is a normalized function of bounded variation with $L(-\infty)=0$ and $L(t)\not\equiv 0$. The operator is proved to be variation diminishing if and only if $L(t)$ is either $\int_{-\infty}^{t}\Lambda(u)du$, where $\Lambda(u)$ is a cumulative Pólya frequency function, or a step function with only one jump. In either case the bilateral Laplace-Stieltjes transform of $L(t)$ is the reciprocal of an entire function of type II in the terminology of Laguerre, Pólya, and I. Schur. In the proof of the direct part of the first result $f(x)$ is allowed to be any bounded measurable function, in the rest of the argument $f(x)$ is continuous. If the bilateral Laplace-Stieltjes transform converges in a strip $-a<\Re(s)<a$, then the variation-diminishing property extends to polynomials and the converse part can also be concluded from a theorem of G. Pólya [J. Reine Angew. Math. **145**, 224–249 (1915), especially p. 231].

E. Hille (New Haven, Conn.).

Polynomials, Polynomial Approximations

***Hall, Tord. On Polynomials Bounded at an Infinity of Points.** Thesis, University of Uppsala, 1950. 47 pp.

Let $\{A_\nu\}$ be a sequence of positive numbers and $\{\lambda_\nu\}=E$ a sequence of distinct complex numbers. We consider the class C of polynomials $P(z)$ satisfying the condition $|P(\lambda_\nu)|\leqq A_\nu$, and form $M(z_0)=\sup|P(z_0)|$. The author makes the assumption that $A_\nu\geqq 1$ and also that $A_\nu^{-1}|\lambda_\nu|^k\to 0$ as $\nu\to\infty$, $k=0, 1, 2, \cdots$. Then $M(z_0)$ is either positive or infinite. The main purpose of the investigation is to find criteria which distinguish these two cases. This is achieved by an appropriate limiting process and by the use of the theory of infinite systems of linear equations. One of the chief results is the following: $M(0)$ is finite if and only if the system $\sum(\xi_\nu/A_\nu)\lambda_\nu^k=0$, $\nu, k=1, 2, 3, \cdots$, has a solution such that $\sum|\xi_\nu|\leqq 1$ and $\sum A_\nu^{-1}\xi_\nu\neq 0$. Further, it is shown that if $\sum D_k$ denotes the complementary set of the closed hull of E, then in each of the regions D_k either $M(z)\equiv\infty$ or $\log M(z)$ is continuous and subharmonic. Finally, various cases with $M(z_0)=\infty$ are established, for instance, the case where the λ_ν are all on a Jordan curve enclosing the origin and $A_\nu\to\infty$. *G. Szegö* (Stanford University, Calif.).

Montel, Paul. Sur les zéros des polynomes associés à un polynome. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 131–136 (1950).

The author investigates the relation between the zeros of the polynomial $\pi(z)=P(z)+iQ(z)$, where $P(z)$ and $Q(z)$ are nth degree real polynomials, and the zeros of the linear combination $P(z)+\lambda Q(z)$, where λ is an arbitrary real number. He proves that, if $\pi(z)$ has p zeros ($n<2p$) in the half-plane $\Im(z)>0$ and $q=n-p$ zeros in the half-plane $\Im(z)<0$, then every polynomial $P(z)+\lambda Q(z)$, λ real, has at least $p-q$ real, distinct zeros. His proof consists in allowing a point ζ to vary continuously along a line from $z=i$ to $z=\lambda$ and observing that no zero of $P(z)+\zeta Q(z)$ can become real until ζ reaches λ at which point nonreal zeros must form conjugate pairs. The author also proves a converse theorem to the effect that, if the polynomial $P(z)+\lambda Q(z)$ has $n-2k$ real zeros for each real λ, then these zeros are all distinct and the polynomial $\pi(z)$ has k zeros in the half-plane $\Im(z)<0$ and the remaining $n-k$ zeros in the half-plane $\Im(z)>0$. When $q=0$, the direct theorem reduces to a result of Hermite and Biehler and the converse theorem reduces to a result of Kakeya [see Fujiwara, Tôhoku Math. J. **9**, 102–108 (1916)]. *M. Marden* (Milwaukee, Wis.).

Markovitch, D. Extension d'un théorème de Hurwitz. Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 113–115 (1949). (Serbian. French summary)

L'équation trinôme $1+x^p+\alpha x^m=0$ possède au moins un zéro dans le cercle $|x+1|\leqq 1$ si p désigne un entier positif impair ($p<m$). *From the author's summary.*

Leonhard, A. Neues Verfahren zur Stabilitätsuntersuchung. Arch. Elektrotechnik **38**, 17–28 (1944).

The author describes a test for stability which is similar in some respects to that of Hurwitz, and, in its reliance on the behavior of a curve, to that of Nyquist. He applies the test to examples, in some of which his method leads to approximate roots of the characteristic equation.

P. Franklin (Cambridge, Mass.).

Parodi, Maurice. Sur la définition des réseaux électriques maillés dont l'équation aux fréquences propres a ses racines à l'intérieur d'un cercle donné. J. Phys. Radium (8) **11**, 141–143 (1950).

The author obtains a condition sufficient to ensure that the equation $|l_{jk}z^2+r_{jk}z+s_{jk}|=0$ has its roots inside a given circle which lies to the left of the imaginary axis and has a real center. The matrices (l_{jk}), (r_{jk}), (s_{jk}) are all assumed positive definite. This result is closely connected with a previous paper of the author [C. R. Acad. Sci. Paris **229**, 1190–1192 (1949); these Rev. **11**, 354].

O. Todd-Taussky (Washington, D. C.).

Zoukhovitsky, S. Sur l'approximation de certaines fonctions rationnelles dans un petit intervalle. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 135–143 (1947). (Ukrainian. Russian and French summaries)

Let $f(x)$ be one of the rational functions

$$(*) \qquad \sum_{i=1}^{h} A_i/(x-a)^i, \quad \sum_{i=1}^{s} A_i/(x-a_i).$$

The author considers the best approximation $E_n[f, h]$ of f by polynomials of degree n in a small interval $(-h, h)$. In the first case the first few terms of the asymptotic development of $E_n[f, h]$ into a power series in h are computed. For the second function (*), the general scheme of computation is indicated and the case $s=2$ is considered in detail.

A. Zygmund (Cambridge, Mass.).

Pollaczek, Félix. Sur une famille de polynômes orthogonaux à quatre paramètres. C. R. Acad. Sci. Paris **230**, 2254–2256 (1950).

Continuing his recent work on orthogonal polynomials, the author determines the system of orthogonal polynomials for the weight function

$$\rho(z)=\frac{(2\sin t)^{2\lambda-1}}{2\pi\Gamma(2\lambda+c)\Gamma(c+1)}e^{\tau(2t-\pi)}|\Gamma(\lambda+c+i\tau)|^2 \times|F(1-\lambda+i\tau, c; c+\lambda+i\tau; e^{2it})|^{-2}.$$

Here z is the variable, the interval is $(-1, 1)$, $z=\cos t$, and τ is an abbreviation for $(a\cos t+b)/\sin t$. The paper gives the generating function, recurrence relation, and proof of orthogonality, under certain conditions on the parameters a, b, c, λ.

A. Erdélyi (Pasadena, Calif.).

Turán, P. Remark on a theorem of Fejér. Publ. Math. Debrecen **1**, 95–97 (1949).

Let $s_n^{(\mu)}=s_n^{(\mu)}(x)$ be the Cesáro sums of the geometrical series, defined by the following recursion:

$$s_n^{(\mu)}=s_0^{(\mu-1)}+s_1^{(\mu-1)}+\cdots+s_n^{(\mu-1)},$$
$$s_n^{(0)}=1+x+\cdots+x^n; \quad \mu=1, 2, \cdots.$$

Fejér has shown [Acta Litt. Sci. Szeged **8**, 89–115 (1937)] that the polynomials

$$A_n(x, \mu)=(d/dx)^\mu s_n^{(\mu)}(x), \qquad n\geqq\mu,$$

are positive on the range $-1<x<1$. By using the generating series of $A_n(x, \mu)$ the author derives a representation of these polynomials in terms of certain ultraspherical polynomials and infers from it the following result. All zeros of these polynomials are on the unit circle $|x|=1$.

G. Szegö (Stanford University, Calif.).

Special Functions

Sansone, Giovanni. Due semplici limitazioni nel campo complesso delle funzioni associate ai polinomi di Tchebychef-Hermite e del termine complementare della loro rappresentazione asintotica. Math. Z. **52**, 593–598. (1950).

The orthonormal sequence of functions

$$f_n(x)=\pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}}e^{\frac{1}{2}x^2}(d^n/dx^n)e^{-x^2}$$

associated with Hermite polynomials satisfies the integral equation

$$f_n(x)=f_n(0)\cos\nu x+\nu^{-1}f_n'(0)\sin\nu x -\nu^{-1}\int_0^x y^2 f_n(y)\sin\{\nu(y-x)\}dy,$$

where $\nu=(2n+1)^{\frac{1}{2}}$. From this equation it is easy to derive an inequality for $f_n(x)$, for real x, by applying Schwarz's inequality to the integral. The author derives an inequality for complex x by estimating the resolvent kernel of this integral equation. By iteration of the integral equation one obtains an asymptotic expansion (for large n). The author estimates the remainder in this asymptotic expansion.

A. Erdélyi (Pasadena, Calif.).

Hodapp, Walter. Über die Hermiteschen Funktionen zweiter Art von reellem und rein imaginärem Argument. Arch. Math. **2**, 186–191 (1950).

Two solutions of the differential equation $y''-xy'+ny=0$, n a nonnegative integer, are the Hermite polynomial

$$H_n(x)=(-1)^n e^{\frac{1}{2}x^2}(d^n/dx^n)e^{-\frac{1}{2}x^2}$$

and the Hermite function of the second kind

$$h_n(x)=(-1)^n e^{\frac{1}{2}x^2}(d^n/dx^n)\int_0^x e^{-\frac{1}{2}(x^2-u^2)}du.$$

The author gives the power series expansion of $h_n(x)$, explicit forms for $n=0, 1, 2, 3$, asymptotic expansions for large x, and some results on the real zeros. He also discusses the behaviour of his functions for imaginary values of the variable. There is no attempt to correlate the theory developed in this paper with the known results on parabolic cylinder functions. *A. Erdélyi* (Pasadena, Calif.).

Palamà, Giuseppe. Funzioni di Laguerre di 2ª specie. Boll. Un. Mat. Ital. (3) **5**, 72–77 (1950).

Laguerre and Hermite functions of the second kind are nonpolynomial solutions of the differential equations satisfied respectively by Laguerre and Hermite polynomials. In this paper the functions of the second kind are defined as

$$l_n^{(\alpha)}(x) = -(2\pi)^{-\frac{1}{2}}\Gamma(\alpha)x^{-\alpha}\,{}_1F_1(-\alpha-n;\,1-\alpha;\,x),$$
$$h_{2n}(x) = (-1)^n 2^{2n+\frac{1}{2}}n!\,x\,{}_1F_1(\tfrac{1}{2}-n;\,\tfrac{3}{2};\,x^2),$$
$$h_{2n+1}(x) = (-1)^{n+1}2^{2n+\frac{1}{2}}n!\,{}_1F_1(-\tfrac{1}{2}-n;\,\tfrac{1}{2};\,x^2),$$

where ${}_1F_1$ is the confluent hypergeometric series. For these functions the author derives a few formulae; all of them are paraphrases of known formulae for the confluent hypergeometric series [cf. also the preceding review].

A. Erdélyi (Pasadena, Calif.).

Mitra, S. C., and Sharma, A. On a generalisation of Weber's parabolic cylinder functions. Proc. Benares Math. Soc. (N.S.) **9**, 25–31 (1947).

Mitra, S. C., and Sharma, A. On a generalisation of Weber's parabolic-cylinder functions. Bull. Calcutta Math. Soc. **41**, 87–91 (1949).

In these two papers the authors introduce a generalisation of the well-known parabolic cylinder functions. On their own showing, this generalised parabolic cylinder function can be expressed in terms of the confluent hypergeometric function, and, in fact, in terms of generalised Laguerre polynomials. Most of the results given in the two papers are transcriptions of known results on generalised Laguerre polynomials, but the proofs are partly new.

A. Erdélyi (Pasadena, Calif.).

Picht, Johannes. Über Integrale von Funktionen, die Produkte Besselscher Funktionen enthalten. Z. Angew. Math. Mech. **29**, 155–157 (1949).

The author investigates the indefinite integrals of $x^{-q}J_m(x)J_n(x)$. He gives explicit formulas for $q=0, 1, 2, 3$ without proof and derives a formula for general q.

A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Tranter, C. J. Legendre transforms. Quart. J. Math., Oxford Ser. (2) **1**, 1–8 (1950).

Let (z, ρ, θ) be cylindrical coordinates. The author studies the problem of determining a function $V(z, \rho)$ satisfying Laplace's equation for $z>0$, and subject, for $z=0$, to the mixed boundary conditions $\partial V/\partial z = f(\rho)$, $\rho<a$; $V=F(\rho)$, $\rho \geqq a$. A transformation is made to oblate spheroidal coordinates $z=a\mu\zeta$, $\rho=a(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}$. For each ζ the solution is expanded in a series of Legendre polynomials in μ with coefficients depending on ζ. An explicit expression is obtained for these coefficients. The formal solution thus obtained is shown to be correct for a certain special choice of f and F. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

***Lyapunov, A. M. Raboty po teorii potenciala. [Works on Potential Theory]. With a biographical sketch by V. A. Steklov.** Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1949. 179 pp.

Sips, Robert. Quelques propriétés extrémales du cercle et de la sphère. Amer. J. Math. **72**, 595–611 (1950).

Let $z(x, y)$ be a solution of $\nabla^2 z = f(z)$ in a ring-shaped domain bounded by the interior curve C_i and the exterior curve C_e; let $z=z_i$ on C_i, $z=z_e$ on C_e, $z_i<z_e$. We assume that $f(z)$ is analytic and positive for $z_i \leqq z \leqq z_e$. Further, we assume that the level sets $z(x, y)=z$ consist of a finite number of mutually exclusive curves with total area $S(z)$ and that for increasing z these curves exhaust the given domain in an increasing manner. The fundamental inequality is as follows:

$$4\pi S dz/dS \leqq \int_{S_i}^{S} f(z)dS + \int_{C_i} (\partial z/\partial n)dS,$$

where $S_i = S(z_i)$. The interior curve might be replaced by a system of curves and some of these curves may degenerate to a point. A similar inequality is obtained in space. This inequality is applied to the following problems: electrostatic capacity, torsional rigidity, fundamental frequency of a membrane, the problem of Liapounoff on the equilibrium of incompressible fluids, etc. We mention the following result (which was recently obtained also by W. R. Wasow in an unpublished paper). Let D be a simply connected domain, $\nabla^2 z = -2$ in D, $z=0$ on the boundary. The maximum z_m of z in this domain is less than the corresponding maximum for a circular domain of equal area.

G. Szegö (Stanford University, Calif.).

Pepper, P. M. The algebraic character of a class of harmonic functions in three variables. Proc. Amer. Math. Soc. **1**, 90–98 (1950).

Recherche du caractère algébrique d'une fonction harmonique de 3 variables,

$$F(x, y, z) = \sum_{n=0}^{\infty} \sum_{\nu=-n}^{n} A_{n,\nu} r^n P_n^{(\nu)}(\cos\theta) e^{i\nu\varphi}$$

à partir des coefficients $A_{n,\nu}$ du développement de F en coordonnées sphériques

$$x = r\cos\theta, \quad y = r\sin\theta\cos\varphi, \quad z = r\cos\theta\cos\varphi.$$

L'auteur utilise un résultat de S. Bergman [Math. Ann. **99**, 629–659 (1928); **101**, 534–558 (1929)] sous la forme suivante: si $u = x + iy(\frac{1}{2}(\zeta+\zeta^{-1})) + z(\frac{1}{2}(\zeta-\zeta^{-1}))$ et si $g(u, \zeta)$ est une fraction rationelle de u, ζ, alors

$$F(x, y, z) = (2i\pi)^{-1}\int_C g(u, \zeta)d\zeta,$$

où C est le circle $|\zeta|=1$, est une fonction harmonique et algébrique. En utilisant $g(u, \zeta) = \sum_{n=0}^{\infty}\sum_{\nu=-n}^{n} a(n, \nu)u^n\zeta^\nu$, une condition suffisante est donnée pour que F soit algébrique. Elle porte sur les nombres

$$a(n, \nu) = 2^n(n+\nu)!(n-\nu)![(2n)!]^{-\frac{1}{2}}i^{-\nu}A_{n,\nu},$$

et elle s'obtient en exprimant au moyen des déterminants de Kronecker $A_0^\mu(\zeta)$ que g est rationelle de u, ζ étant un paramètre, et an écrivant que le coefficient $A_0^\mu\{m\}$ de ζ^m dans $A_0^\mu(\zeta)$ est nul $(-\mu^2-\mu \leqq m \leqq \mu^2+\mu)$. *P. Lelong.*

Bergman, Stefan. On solutions with algebraic character of linear partial differential equations. Trans. Amer. Math. Soc. **68**, 461–507 (1950).

L'auteur introduit et étudie une correspondance entre toute fonction harmonique réelle $h(x, y, z)$ au voisinage de l'origine (dans l'espace à 3 dimensions) et une fonction χ analytique de deux variables complexes $Z = \frac{1}{2}(iy+z)$ et $Z^* = \frac{1}{2}(iy-z)$, qui coincide avec le prolongement de u pour

variables complexes sur le cône $x^2+y^2+z^2=0$. Il introduit un intermédiaire f à l'aide de la formule de représentation de Whittaker, et les opérateurs transformant χ en f et f en h. Cela lui permet d'étudier les fonctions harmoniques algébriques sur les variétés à 3 dimensions analogues aux surfaces de Riemann. Puis ces considérations sont prolongées aux vecteurs harmoniques, c'est-à-dire de div et rot nuls. Enfin sont étudiés des opérateurs transformant les fonctions harmoniques en solutions de $\Delta\Phi+C\Phi=0$ où C est fonction entière de $x^2+y^2+z^2$, ce qui permet une étude des solutions qui sont fonctions algébriques de y et z sur l'espace $x^2+y^2+z^2=0$. *M. Brelot* (Grenoble).

Ugaeri, Tadashi. On the harmonic prolongation. J. Math. Soc. Japan **1**, 262–265 (1949).

A proof is given for the known result that if $v(x, y)$ is harmonic in the open upper half of the unit circle, and continuous on its closure, and the normal derivative of $v(x, y)$ vanishes along the x-axis, then $v(x, -y)=v(x, y)$ defines a harmonic continuation of $v(x, y)$ throughout the unit circle. It is pointed out that since the partial derivative with respect to x is not assumed to be continuous on the x-axis, the usual proof by means of Green's formula is not applicable. We note that the present result can be obtained perhaps most easily by observing that the harmonic function $\partial v/\partial y$ can be continued harmonically across the x-axis, whence by partial integration the same result holds for $v(x, y)$.

E. F. Beckenbach (Los Angeles, Calif.).

Walsh, J. L. The location of critical points of harmonic functions. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 61–65 (1950).

In a study of the critical points of a harmonic function, the author applies the following representation for superharmonic functions due to F. Riesz [Acta Math. **54**, 321–360 (1930)]. If there exists a function which in a regular region R is harmonic and inferior to a function $u(z)=u(x, y)$ which is continuous in R, then $u(z)$ can be represented in R in the form $u(z)=\int_R g(z, t, R)d\mu(e)+h(z)$, $0\leqq\mu(e)$, where $g(z, t, R)$ is Green's function for R with pole at z, $h(z)$ is harmonic in R, and the integration over R is taken with respect to a suitably chosen positive additive set function $\mu(e)$ defined for t on all open sets e having their closures in R. By means of this representation and some methods of his earlier papers, the author is able to establish the following theorem. Let R be the interior of a Jordan curve C, let B be a closed set in R which together with C bounds a subregion R' of R, and let $u(z)$ be harmonic in R', superharmonic in R, continuous in the closed neighborhood of C in $R+C$, and zero on C; then all the critical points of $u(z)$ in R' lie in the smallest non-Euclidean convex set of R containing B. This theorem is a generalization of an earlier result by the author [Bull. Amer. Math. Soc. **45**, 462–470 (1939)]. A second theorem proved by use of F. Riesz' representation is the following. Let π_1 and π_2 be the upper and lower half-planes, let R be the region $|z|<1$, let R' be a subregion of R bounded by the unit circle C and by a closed set B in R not intersecting the real axis A, and let $u(x, y)$ be harmonic in R', continuous in the closed neighborhood of C in $R+C$, zero on C, superharmonic in $R\cdot\pi_1$ and subharmonic in $R\cdot\pi_2$; then the origin is not a critical point of $u(x, y)$. This theorem is also a wide generalization of the results which can be proved without using the above representation.

M. Marden (Milwaukee, Wis.).

Brelot, M. Remarque sur le prolongement fonctionnel linéaire et le problème de Dirichlet. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 150–152 (1950).

Let Ω be a bounded domain in Euclidean space R^p, $p\geqq 2$, with boundary Ω^*. It is known that to each real function f, defined and continuous on Ω^*, there corresponds the Wiener solution H_f to the generalized Dirichlet problem for the pair $[\Omega, f]$. Keldych has shown [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 308–309 (1941); these Rev. **6**, 155] that H_f is unique in the following sense. The classical solution to the Dirichlet problem (with the partial ordering usually subsisting among real functions) yields a linear, homogeneous, nondecreasing mapping of a set J of certain of the functions f into the partially ordered set E of real functions harmonic in Ω; then Keldych's result states that there is a unique extension of this mapping to all the functions f with the preservation of linearity, homogeneity and positivity.

The present author noted the resemblance of Keldych's result to the classical Hahn-Banach extension theorem; he uses some of his earlier results and a modification of Banach's proof of the extension theorem to obtain the following result. The solutions of the classical problem of Dirichlet which take on values not greater than f on Ω^* have an upper envelope $\underline{D}_f$ which is finite, continuous and subharmonic, and whose least harmonic majorant $\underline{D}_f$ is the supremum of the solutions considered in the set E. Similarly, the solutions not less than f on Ω^* have a lower envelope $\bar{D}_f$ which is finite, continuous and superharmonic, and whose greatest harmonic majorant $\bar{D}_f$ is the infimum of these solutions. Finally, $\underline{D}_f=\bar{D}_f=H_f$. The author concludes the note with suggestions for further research; for example, the possible extension of the preceding to suitably ordered vector spaces, in the spirit of an (uncited) recent work of Choquet.

M. O. Reade (Ann Arbor, Mich.).

Bonsall, F. F. On generalized subharmonic functions. Proc. Cambridge Philos. Soc. **46**, 387–395 (1950).

The notion of subharmonic function can be generalized by replacing the dominating family of harmonic functions by the family of solutions of a more general elliptic partial differential equation, or even by a quite general family having suitable properties (see, for instance, Tautz [Math. Ann. **117**, 694–726 (1941); **118**, 733–770 (1943); these Rev. **3**, 126; **6**, 3]). For the elliptic differential case, the author points out that subfunctions are characterized by integral inequalities and, if they are of class C'', by differential inequalities; these reduce to the familiar inequalities in the subharmonic case. It is shown that certain functions of a single variable, like $\sup_{-\infty<y<\infty} f(x, y)$ and $\limsup_{y\to\infty} f(x, y)$, related to generalized subharmonic functions $f(x, y)$, are themselves generalized convex functions, in the same way that corresponding functions related to subharmonic functions are convex functions. Applications are made in the theory of analytic functions of a complex variable. Thus it is shown that if $f(z)$ is analytic and schlicht in the half-plane $x=\Re(z)>0$, then $|f'(z)|$ is a superfunction relative to the family of solutions of $\partial^2\varphi/\partial u^2+\partial^2\varphi/\partial v^2-9/\varphi^2=0$.

E. F. Beckenbach (Los Angeles, Calif.).

Differential Equations

Picone, Mauro. Ulteriore analisi quantitativa delle soluzioni di talune equazioni differenziali ordinarie. Ann. Mat. Pura Appl. (4) **28**, 195–203 (1949).

This paper deals with systems of ordinary differential equations, in the real domain, such that when a system is written in the normal form the right-hand members of the equations are polynomials in the unknowns with coefficients which are quasi-continuous functions of the independent variable. The object of the work is to obtain improved estimates of such things as the length of an interval in which the existence of a solution is assured, the lengths of the intervals separating the zeros of one of the functions composing a solution, etc. The complexity of the results precludes any full restatement of them here. The following two simple theorems are indicative of the kind of results which are obtained. (A) Let $f_0(t)$, $f_1(t)$ be quasi-continuous functions defined over an interval I; and let $|f_i| \leqq a_i$, where a_0 and a_1 are constants. Let $y(t)$ be a solution of the equation $y''=f_0(t)y+f_1(t)y'$ such that $y(t_0)\neq 0$, where t_0 is a particular point of I. Set $x_0=y'(t_0)/y(t_0)$ and

$$\lambda^*(x_0)=[a_0+a_1|x_0|+|x_0|^2]^{\frac{1}{2}}.$$

Then $y(t)$ has no zeros in the common part, J, of I and the interval $|t-t_0| \leqq [2\lambda^*(x_0)+a_1+2|x_0|]^{-1}$. Also, for values of t in J we have the relation

$$\left|\frac{y'(t)}{y(t)}-\frac{y'(t_0)}{y(t_0)}\right| \leqq [2\lambda^*(x_0)+a_1+2|x_0|]\lambda^*(x_0)|t-t_0| \leqq \lambda^*(x_0).$$

(B) Under the same conditions as in (A), if $y(t)$ is a solution which is not identically zero, the number of zeros of $y(t)$ in a finite subinterval (a, b) of I is less than $1+(b-a)[a_0^{\frac{1}{2}}+\frac{1}{2}a_1]$.

L. A. MacColl (New York, N. Y.).

Tartakovskiĭ, V. Explicit formulas for the local expansions of solutions of a system of ordinary differential equations. Doklady Akad. SSSR (N.S.) **72**, 633–636 (1950). (Russian)

The author describes a new method for solving in explicit form (locally) a system (1) $dx_i/dt=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, where the f_i are holomorphic at the origin. Let the monomials $x_1^{k_1} \cdots x_n^{k_n}$, $k_i \geqq 0$, be viewed as the (countable) components of a row-vector y. Then (1) is equivalent to the linear equation (2) $dy/dt=yA$, where A is an infinite constant matrix such that in each row all but a finite number of elements are zero. The solution is based on the lemma: If $(A^\nu)^{(q)}_{(p)}$ is the matrix of the elements of A^ν in the rows $p_1, \cdots, p_n$ and columns $q_1, \cdots, q_n$ and if $p=p_1+\cdots+p_n$, $q=q_1+\cdots+q_n$, then there exist positive constants N, r such that for every positive integer ν,

$$|(A^\nu)^{(q)}_{(p)}| \leqq qr^q(2/r)^p N^\nu \cdot (\nu-1)!.$$

The explicit form of the solution of (2) is $y=y_0e^{At}$, when y_0 is the initial value of y. No proof of the lemma is given and the derivation of the solution is only outlined. The author states that his method yields a new proof of Cauchy's existence theorem. *S. Lefschetz* (Princeton, N. J.).

Popov, B. S. Sur la condition d'intégrabilité de Karamata de l'équation de la balistique extérieure. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2**, 249–263 (1949). (Serbian. Russian and French summaries)

A generalised form of d'Alembert's condition for integration of the ballistic differential equation,

$$(y+Ae^{ns}+Be^{-ns}+R)y'+y^2-1=0,$$

was published by Karamata as a problem [Bull. Soc. Math. Phys. Serbie **1**, no. 2, 72–73 (1949)]. The author proves this result by transforming the above equation into a Legendre or a hypergeometric equation.

W. S. Jardetzky (New York, N. Y.).

Viguier, Gabriel. Canonisation géométrique de l'équation d'Abel. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 145–148 (1949). (French. Croatian summary)

Viguier, Gabriel. L'oscillateur harmonique à n dimensions. Experientia **6**, 222 (1950).

The n-dimensional oscillator problem is related to a Riccati equation and the associated geometry of a family of plane curves. *P. Franklin* (Cambridge, Mass.).

Makovska, G. Calcul des réservoirs cylindriques avec une paroi d'épaisseur variable. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 145–152 (1947). (Ukrainian. Russian and French summaries)

This paper gives the computations of the horizontal displacements of an element of a filled, cylindrical reservoir with vertical axis. It is assumed that the thickness of the walls of the reservoir is given by $\delta(x)=x^2+c_1x+c_2$, where x is the distance from the top of the reservoir. The problem is said to be equivalent to the solution of the fourth order differential equation (1) $(d^2/dx^2)[f(x)d^2w/dx^2]+\varphi(x)w=\gamma(x)$, where w is the horizontal displacement of an element of the wall, $f(x)=E\delta^3(x)/12(1-\mu^2)$, E is Young's modulus, μ is Poisson's ratio, $\varphi(x)=E\delta(x)/R^2$, R is the radius of the cylinder, and γ is the density of the medium filling the cylinder. The equation (1) is transformed into a system of two differential equations of the second order. The solution of this system is reduced by d'Alembert's method to that of the hypergeometric equation of Gauss. The final results are given in terms of hypergeometric functions.

H. P. Thielman (Ames, Iowa).

Kovalenkov, V. I. Decomposition of equations describing electromagnetic processes in local circuits. Avtomatika i Telemehanika **8**, 3–19 (1947). (Russian)

A detailed discussion of systems of linear differential equations with constant coefficients arising in connection with various electromagnetic circuits. *S. Lefschetz.*

El'šin, M. I. Qualitative solution of a linear differential equation of the second order. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 155–158 (1950). (Russian)

The author points out that his "phase method," which consists of setting $x=y\exp(\int_{t_0}^{t}(\theta-\frac{1}{2}p)dt_1)$ in the differential equation $x''+p(t)x'+q(t)x=0$, obtaining $(ky')'+gy=0$, where k and g depend upon θ (which is to be chosen cleverly), may be used to obtain many of the known results concerning the qualitative behavior of the solutions of (1). Examples and references are given. *R. Bellman.*

Aĭzerman, M. A. On the convergence of processes of automatic regulation after large initial deflections. Avtomatika i Telemehanika **7**, 148–167 (1946). (Russian)

Aĭzerman, M. A. On taking account of nonlinear functions of several variables in the investigation of the stability of a system of automatic regulation. Avtomatika i Telemehanika **8**, 20–29 (1947). (Russian)

In the first paper, the author considers the problem of determining the stability of the solutions of systems of

differential equations of the form

$$dx_k/dt=\sum_{j=1}^{N}a_{kj}x_j+f(x_k),\quad dx_i/dt=\sum_{j=1}^{N}a_{ij}x_j,$$

$i=2, 3, \cdots, N$, where k is some fixed integer between 1 and N, the a_{ij} are constants, and $f(0)=0$. Using the second method of Lyapunov, various criteria are obtained which are applied to some practical problems. In the second paper, he extends his methods to cover the general case where $dx_i/dt=f_i(x_1, x_2, \cdots, x_N)$, and gives further applications.

R. Bellman (Stanford University, Calif.).

***Malkin, I. G. Metody Lyapunova i Puankare v teorii nelineinyh kolebanii. [The Methods of Lyapunov and Poincaré in the Theory of Nonlinear Oscillations].** OGIZ, Moscow-Leningrad, 1949. 244 pp.

Those who have perused Minorsky's "Introduction to nonlinear mechanics" [Edwards, Ann Arbor, Mich., 1947; these Rev. 8, 583] are well acquainted with the fact that this subject has been treated in the Soviet Union by two distinct schools, one in Moscow and the other in Kiev. The work of the first, centered around the Institute of Oscillations, is fairly well represented by the material in the "Theory of oscillations" by Andronow and Chaikin [Moscow, 1937; editorially revised English edition, Princeton University Press, Princeton, N. J., 1949; these Rev. 10, 535]. In the main, the investigations of this school have been based upon the mathematically strictly correct work of Poincaré [Les méthodes nouvelles de la mécanique céleste, Gauthier-Villars, Paris, 1892–1899] and of Lyapunov [Problème général de la stabilité du mouvement, Princeton University Press, Princeton, N. J., 1947; these Rev. 9, 34]. The procedure is dominated on the analytical side by the small parameter method with approximations known to be convergent. Parallel investigations took place at Kiev, where Krylov and Bogoljubov developed intensively the "pre-Poincaré" methods of the astronomers Gylden and Lindstedt based on approximations, generally divergent according to Poincaré, but none the less of much practical value. This second mode of attack is reasonably well represented in the monograph by Krylov and Bogoljubov, "Introduction to non-linear mechanics" [Kiev, 1937; much abridged edition by the reviewer, Princeton University Press, Princeton, N. J., 1943; these Rev. 4, 142].

The author of the present monograph is one of the foremost adherents of the Moscow school. The treatment assumes the fundamental existence theorems and deals exclusively with the periodic solutions of systems

$$dx_s/dt=X_s(t, x_1, \cdots, x_r, \mu), \tag{1}$$

where X_s is analytic in the x_i and μ, for the x_i in a certain region and μ small, and is continuous and periodic with period 2π in t (but may fail to contain t). Given a "generating" periodic solution $f_s(t)$ of (1) for $\mu=0$, the Poincaré theory for the existence of neighboring solutions is discussed and considerably extended. The Lyapunov stability criteria for periodic solutions are thoroughly discussed and applied to the problem. The results of the second part of Lyapunov's paper mentioned above, dealing with periodic solutions, are taken up with particular emphasis on Lyapunov systems (the author's terminology), i.e., reducible to the form

$$\begin{cases}\dfrac{dx}{dt}=-\lambda y+X, & \dfrac{dy}{dt}=\lambda x+Y,\\ \dfrac{dx_s}{dt}=\sum b_{si}x_i+X_s,\end{cases} \tag{2}$$

where λ and the b_{si} are constants, and X, Y, X_s do not contain t, are analytic at the origin, and begin with terms of degree at least two. Furthermore it is assumed that (2) has an integral

$$H=x^2+y^2+W(x_1, \cdots, x_m)+S(x, y, x_1, \cdots, x_m)=\text{constant},$$

where W is a quadratic form and S is analytic at the origin and begins with terms of degree at least three. Lyapunov systems possess a family of periodic solutions. Systems very near to those of Lyapunov are investigated for their periodic solutions. A goodly number of examples, drawn mainly from differential equations of the second order, are carefully treated throughout the monograph. *S. Lefschetz.*

Malkin, I. G. Oscillations of quasilinear systems with a nonanalytic characteristic of nonlinearity. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 13–22 (1950). (Russian)

In a recent monograph [see the preceding review] the author has investigated the quasi-harmonic oscillations of a system $\ddot{x}+k^2x=\mu F(t, x, \dot{x})$, where F is periodic and of period 2π in t with a related Fourier series, and is analytic in x, $\dot{x}$ in a certain region. This investigation is here extended to the case where F_x, $F_{\dot{x}}$ satisfy a Lipschitz condition.

S. Lefschetz (Princeton, N. J.).

***Wendel, James G. Singular perturbations of a van der Pol equation.** Contributions to the Theory of Nonlinear Oscillations, pp. 243–290. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N. J., 1950. $4.00.

The author gives sufficient conditions on f, g, and e so that solutions of the differential equation $\epsilon\ddot{x}+f(x)\dot{x}+\epsilon g(x)=e(t)$ tend as $\epsilon\to 0$ to certain solutions of a degenerate equation $f(y)\dot{y}=e(t)$. Here $e(t)$ is periodic. As an application he shows the equation $\epsilon\ddot{x}+(x^2-1)\dot{x}+\epsilon x=b\lambda\cos\lambda t$ satisfies his requirements if $b>\frac{2}{3}$. [When $b<\frac{2}{3}$ the equation has extremely complicated behaviour, as was shown by Cartwright and Littlewood, J. London Math. Soc. 20, 180–189 (1945); these Rev. 8, 68.] *N. Levinson* (Cambridge, Mass.).

Žabotinskiĭ, M. E. Auto-oscillating systems with two degrees of freedom in the case of multiple frequencies. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 20, 421–426 (1950). (Russian)

The author applies the standard Poincaré-Lyapunov methods to a pair of quasi-harmonic oscillations reducible to $\ddot{\xi}+\xi=\mu f(\xi, \dot{\xi}, \eta, \dot{\eta})$, $\ddot{\eta}+n^2\eta=\mu g(\xi, \dot{\xi}, \eta, \dot{\eta})$, where μ is small and f, g are holomorphic in their arguments in the vicinity of a periodic solution of the system corresponding to $\mu=0$. It is assumed that n is an integer which causes departures from the general case. Solutions are discussed which are of the form $\xi=R\cos t+x$, $\eta=P\cos nt+Q\sin nt+y$, where $x, y\to 0$ with μ, and which have a period $2\pi+\tau$, where again $\tau\to 0$ with μ. Explicit equations are obtained, the stability is discussed, and an application is made to an electric system corresponding to $n=3$. *S. Lefschetz* (Princeton, N. J.).

Soroka, Walter W. Free periodic motions of an undamped two-degree-of-freedom oscillatory system with nonlinear unsymmetrical elasticity. J. Appl. Mech. 17, 185–190 (1950).

Some periodic solutions, obtained numerically, are given for the system $I_1\ddot{\theta}_1+T_1-T_2=0$, $I_2\ddot{\theta}_2+T_2=0$, where $T_1=k_1\theta_1$, $T_2=k_2(\theta_2-\theta_1)$ for $\theta_2-\theta_1<c$, $T_2=k_3(\theta_2-\theta_1-c)+k_2c$ for $\theta_2-\theta_1\geqq c$ (I_1, I_2, k_1, k_2, k_3, c are positive constants, to which various numerical values are assigned).

G. E. H. Reuter (Manchester).

***Langenhop, C. E., and Farnell, A. B. The existence of forced periodic solutions of second order differential equations near certain equilibrium points of the unforced equation.** Contributions to the Theory of Nonlinear Oscillations, pp. 291–312. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N. J., 1950. $4.00.

The authors show the existence of a small periodic solution for the equation $\ddot{x}+a\dot{x}+x+\frac{1}{2}x^2=\frac{1}{2}k\cos\omega t$, $a>0$, and k small. More general forms of the equation are considered also. [The topological methods used are longer and yield less information than an analytic approach.]

N. Levinson (Cambridge, Mass.).

Miller, J. C. P. On the choice of standard solutions for a homogeneous linear differential equation of the second order. Quart. J. Mech. Appl. Math. 3, 225–235 (1950).

For large values of $|I(x)|$ the solutions of the differential equation $(d^2y/dx^2)+I(x)y=0$, with real $I(x)$, can be asymptotically approximated by functions of exponential or oscillatory type, depending on whether $I(x)$ is negative or positive. Using these asymptotic expressions, the paper discusses the problem of choosing, in an interval where $|I(x)|$ is not too small, a pair of linearly independent solutions that is well suited for theoretical or numerical investigations. If $I(x)<0$, it is desirable to use two solutions whose ratio varies rapidly with x. If $I(x)>0$, one will try to find two solutions that behave as nearly as possible like trigonometric functions with phase difference $\pi/2$. The paper discusses methods for the determination of such solutions by means of a geometric interpretation of each pair of solutions as a curve in three dimensions.

W. R. Wasow.

***Wasow, Wolfgang. On the construction of periodic solutions of singular perturbation problems.** Contributions to the Theory of Nonlinear Oscillations, pp. 313–350. Annals of Mathematics Studies, no. 20. Princeton University Press, Princeton, N. J., 1950. $4.00.

The author shows that the work of Volk [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 559–574 (1946); Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 433–444 (1947); **12**, 29–38 (1948); these Rev. **8**, 330; **9**, 185, 588] contains a serious error. The author uses the formal scheme of Volk in combination with ideas of an earlier paper [J. Math. Physics **23**, 173–183 (1944); these Rev. **6**, 86] and thereby obtains a rigorous theory. He considers the nth order differential equation $\epsilon^k x^{(n)}=F(x, x', \cdots, x^{(m)}; t; \epsilon)$ where $m<n$, $k>0$, and the right side is analytic. It is shown that solutions of the differential equation for small ϵ are, under certain conditions, related to solutions of the degenerate case where $\epsilon=0$. It is assumed F is periodic in t.

N. Levinson (Cambridge, Mass.).

Hartman, Philip, and Wintner, Aurel. On the essential spectra of singular eigenvalue problems. Amer. J. Math. **72**, 545–552 (1950).

The differential equation

$$(1)\qquad \frac{d}{dt}\left(p(t)\frac{dx}{dt}\right)+(q(t)+\lambda)x=0,$$

with $p(t)$ and $q(t)$ real and continuous, and with $p(t)>0$, is called of the "Grenzpunkt" type, if it possesses, for some real λ, solutions $x(t)$ that are not of type $L^2(0,\infty)$, i.e., for which $\int_0^\infty x^2(t)dt$ is not finite. Denote by S_α the λ-set representing the spectrum of the eigenvalue problem defined by (1), the condition that $x(t)$ be of type $L^2(0,\infty)$, and the boundary condition $x(0)\cos\alpha+p(0)x'(0)\sin\alpha=0$. H. Weyl proved [Math. Ann. **68**, 220–269 (1910)] that the set S' of the accumulation points of S_α is independent of α. This set S' is called the essential spectrum of (1) by the authors. The following theorem concerning S' is proved. A real $\lambda=\lambda_0$ is in the essential spectrum of (1) if and only if there exists a real continuous function $g(t)$ of class $L^2(0,\infty)$ for which the nonhomogeneous differential equation $(px')'+(q+\lambda_0)x=g$ does not possess any solution of class $L^2(0,\infty)$. This theorem characterises each point of S' without reference to S_α or to the whole set S'. The difficult part of the proof is the sufficiency of the condition which requires the explicit construction of a function $g(t)$ for which the nonhomogeneous equation has no solution of class $L^2(0,\infty)$.

W. Wasow (Los Angeles, Calif.).

Ledinegg, E. Über ein dem klassischen Minimumproblem homogener Differentialgleichungen vom Sturm-Liouvilleschen Typus zugeordnetes Variationsprinzip. Acta Physica Austriaca **3**, 273–276 (1949).

On donne l'extension de la transformation de Friedrichs aux problèmes isopérimètriques du calcul des variations:

$$\int_{x_0}^{x_1} f(x, y, y')dx=\text{minimum},\quad \int_{x_0}^{x_1} g(x, y)dx=\text{constante},$$

$$y(x_0)=y(x_1)=0.$$

Comme application, en posant $f=\alpha(x)y'^2$, $g=\rho(x)y^2$, on déduit que le problème de Sturm-Liouville:

$$[\alpha(x)y']'+J\rho(x)y=0,\quad y(x_0)=y(x_1)=0,$$

se transforme dans le suivant:

$$[(\rho(x))^{-1}p']'+J(\alpha(x))^{-1}p=0,\quad p'(x_0)=p'(x_1)=0;$$

les deux problèmes ont donc les mêmes valeurs propres (en excluant la valeur $J=0$ du deuxième problème).

A. Ghizzetti (Rome).

Pokrass, M. P. Operational solution of telegraph equations for nonzero initial and boundary conditions. Ėlektričestvo **1950**, no. 3, 60–64 (1950). (Russian)

The telegraph equations,

$$-\partial u/\partial x=Ri+L\partial i/\partial t,\quad -\partial i/\partial x=Gu+C\partial u/\partial t,$$

are solved in general through straightforward use of the Laplace transformation, the solution being left in transformed form. The solution is completed in three special examples, in each of which $G=L=0$.

R. E. Gaskell.

Pignedoli, Antonio. Sugli autovalori di alcune equazioni differenziali della fisica matematica. Atti Sem. Mat. Fis. Univ. Modena **3**, 265–287 (1949).

On considère le calcul de la plus petite valeur propre λ_1 du problème: $\Delta_2 u+\lambda u=0$ dans S, $du/dn-ku=0$ sur la frontière de S, où S est un domaine de l'espace (ou du plan). La question est résolue en supposant que S soit un parallélépipède (ou un rectangle), une sphère (ou un cercle); la méthode suivie est celle classique (séparation des variables). On donne aussi des limitations pour λ_1 en employant le théorème: si $S\subset S'$ on a $\lambda_1(S)>\lambda_1(S')$.

A. Ghizzetti (Rome).

Tihonov, A. N. On the third boundary problem for an equation of parabolic type. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 193–198 (1950). (Russian)

The author seeks a solution of the equation $u_{xx}=u_t$ $(0\leqq x\leqq l, t\geqq 0)$ which is zero initially and satisfies the bound-

ary conditions $-u_x+a_1u=f(t)$ for $x=0$ and $u_x+a_2u=0$ for $x=l$. This problem is reduced to the study of a static electric field in a homogeneous medium. The solution is represented, as is usual in the method of reflection, by a series of functions, each of which satisfies the above differential equation and initial condition. In addition, the even numbered functions of this series are determined for $x\geqq 0$ and satisfy the above condition at $x=0$; the odd numbered functions satisfy a modified condition at $x=l$. The method of construction of these two sequences of functions is given and the convergence of the series representing $u(x, t)$ is shown to be particularly rapid. The solution of the heat flow problem for a semi-infinite line is given in terms of the solution of the first boundary value problem by means of an integral operator. This gives the solution in a form which is convenient for calculation for small values of t.

C. G. Maple (Washington, D. C.).

Tomilov, E. D. Solution of the equation of heat conduction for hollow and solid cylinders with mixed boundary conditions. Tomsk. Gos. Univ. Učenye Zapiski **8**, 125–139 (1948). (Russian)

A solution of the heat equation $(D_\rho+D)U=0$, with $D_\rho=(\rho^{-1}\partial/\partial\rho)(\rho\partial/\partial\rho)$, $D=\partial^2/\partial\xi^2$, is found in the form $T=\sum_{i=0}^{n}[F_i(\rho)D^i\theta^0(\xi)+G_i(\rho)D^i\pi^0(\xi)]$. This is accomplished by setting $F_0(\rho)=1$, $G_0(\rho)=\ln\rho$, and requiring that

$$D_\rho F_{i+1}+F_i=0, \quad D_\rho G_{i+1}+G_i=0$$

$(i=1, 2, \cdots)$. Then $\theta^0(\xi)$ and $\pi^0(\xi)$ are chosen so that T satisfies conditions of the form $\chi\partial T/\partial\rho+T=\theta(\xi)$ at $\rho=\rho_0$ and at $\rho=1$. Finally, an auxiliary solution T', composed of terms satisfying the corresponding homogeneous conditions at ρ_0 and 1, is added to T, and its coefficients adjusted so that $U=T+T'$ also satisfies similar boundary conditions at $\xi=\pm\xi_1$. Nonstationary temperature problems are solved similarly by using $(D_\rho+D_\tau)U=0$, where $D_\tau=\partial^2/\partial\xi^2-\partial/\partial\tau$.

R. E. Gaskell (Ames, Iowa).

Germay, R. H. J. Sur certains systèmes linéaires d'équations aux dérivées partielles du second ordre. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 67–73 (1949).

Replacing the author's notation by one using vectors, the system of equations considered can be written in the form (1) $\partial^2 z/\partial x\partial y=a\partial z/\partial x+b\partial z/\partial y+cz+f$. Here $z=(z_1, \cdots, z_p)$ stands for a vector while a, b, c are square matrices and f is a vector whose elements are functions of x and y. In a previous paper [Mem. Soc. Roy. Sci. Liège (3) **14**, no. 5 (1927), p. 20], the author showed that the solutions z of (1) which vanish for $x=0$ and $y=0$ can be expressed with the help of a matrix of "Riemann functions." In the present note it is observed that the system of Riemann functions in turn determines the coefficient matrices a, b, c.

F. John (Los Angeles, Calif.).

Germay, R. H. J. Sur certains systèmes linéaires d'équations aux dérivées partielles du troisième ordre. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 148–154 (1949).

The results established by the author in a previous note [see the preceding review] can be extended to systems of third order equations of the form

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=A\frac{\partial^2 u}{\partial y\partial z}+B\frac{\partial^2 u}{\partial z\partial x}+C\frac{\partial^2 u}{\partial x\partial y}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+F\frac{\partial u}{\partial z}+Hu+\Phi,$$

where u, Φ are vectors and A, B, C, $\cdots$, H are square matrices with elements depending on x, y, z. The matrix of Riemann functions of the system again determines the coefficient matrices A, B, $\cdots$, H. *F. John.*

Integral Equations

Bruwier, L., et Franckx, E. Sur l'application d'une méthode d'approximations successives à la résolution d'une système d'équations intégrales. Bull. Soc. Roy. Sci. Liége **18**, 486–497 (1949).

The problem here considered is an existence theorem for the system of integral equations $y_i=y_i^0+\int_{x_0}^{x} f_i(t, y_1, \cdots, y_n)dt$, $i=1, \cdots, n$. The method used follows the ideas introduced by Cerassia [same Bull. **16**, 84–93 (1947); these Rev. **9**, 434] and Germay [ibid. **15**, 442–446, 510–513 (1946); **16**, 2–6, 119–125 (1947); these Rev. **9**, 434, 435] and Franckx [ibid. **17**, 281–286, 308–312 (1948); these Rev. **10**, 711]. The proof of the convergence is made by the introduction of a new notion, that of order of an iterate.

I. A. Barnett (Cincinnati, Ohio).

Germay, R. H. Sur des fonctions généralisant les noyaux itérés des équations intégrales de Volterra, de seconde espèce, à deux variables indépendantes. Bull. Soc. Roy. Sci. Liège **19**, 2–8 (1950).

The author considers Volterra integral equations of the second kind in two variables, viz.,

$$\varphi(x, y)=f(x, y)+\lambda\left[\int_{x_0}^{x}d\xi\int_{y_0}^{y}G(x, y; \xi, \eta)\varphi(\xi, \eta)d\eta+\int_{x_0}^{x}H(x, y; \xi)\varphi(\xi, y)d\xi+\int_{y_0}^{y}K(x, y; \eta)\varphi(x, \eta)d\eta\right].$$

By writing the solution in the usual way as a power series in λ, he defines iterates of G, H, and K which reduce to the ordinary iterates of Volterra for an equation in one variable. The recurrence relations for these generalized iterates are given. The same procedure applies to Volterra equations in any number of variables. *I. A. Barnett.*

Parodi, Maurice. Sur une propriété des équations intégrales de Volterra à n variables. C. R. Acad. Sci. Paris **230**, 2252–2253 (1950).

The solution of the Volterra equation

$$f(x_1, \cdots, x_n)+\lambda\int_0^{x_1}\cdots\int_0^{x_n}K(x_1-y_1, \cdots, x_n-y_n)\times f(y_1, \cdots, y_n)dy_1\cdots dy_n=g(x_1, \cdots, x_n)$$

is expressed in terms of the solution when

$$g=x_1^{m_1-1}\cdots x_n^{m_n-1}\exp\left(\sum_1^n \alpha_k x_k\right).$$

This generalises formulae found by the author [same C. R. **217**, 523–525 (1943); these Rev. **6**, 158] and Delerue [same C. R. **229**, 807–808 (1949); these Rev. **11**, 183].

G. E. H. Reuter (Manchester).

De Bruijn, N. G. On some Volterra integral equations of which all solutions are convergent. Nederl. Akad. Wetensch., Proc. **53**, 813–821 = Indagationes Math. **12**, 257–265 (1950).

The author wishes to find conditions on the kernel of the integral equation $f(x)=\int_0^1 K(x, t)f(x-t)dt$ which will entail

the existence of $\lim_{x\to\infty} f(x)$ for every solution $f(x)$. Attention is restricted to solutions which are measurable, bounded on any finite interval, and "regular," i.e., if $m \leqq f(x) \leqq M$ holds almost everywhere on $a-1 \leqq x \leqq a$, then it also holds for $x>a$. Kernels for which $\lim_{x\to\infty} f(x)$ exists for solutions belonging to this class are called "stabilizing." Several sufficient conditions are given for this to be true, of which we quote one: $K(x,t)$ is stabilizing if $K(x,t) \geqq k(t)$ for $x \geqq 0$, $0 \leqq t \leqq 1$, where $k(t)$ is a nonnegative integrable function over $0 \leqq t \leqq 1$ with $\int_0^1 k(t)dt > 0$. Application is made to an integral equation occurring in a problem in prime numbers.

J. V. Wehausen (Providence, R. I.).

Heins, Albert E. Sur les couples d'équations intégrales. C. R. Acad. Sci. Paris **230**, 1732–1734 (1950).

The dual integral equation $\int_0^\infty f(u)J_0(ru)du=1$, $0<r<1$, $\int_0^\infty uf(u)J_0(ru)du=0$, $r>1$, considered by E. C. Titchmarsh [Introduction to the Theory of Fourier Integrals, Oxford University Press, 1937, § 11.16] and I. W. Busbridge [Proc. London Math. Soc. (2) **44**, 115–129 (1938)] is solved by a more direct method. Writing $\int_0^\infty f(u)J_0(ru)du = g_1(r)$ (unknown for $r>1$), $\int_0^\infty uf(u)J_0(ru)du = g_2(r)$ (unknown for $0<r<1$), a relation between the Mellin transforms G_1, G_2 of g_1, g_2 is found, and then G_1, G_2 are determined by the Wiener-Hopf factorisation technique. Hence the Mellin transform of f, and f itself, can be found. *G. E. H. Reuter.*

Rubinowicz, A. Solution of the system of integral equations in Dirac's one-electron problem in momentum representation. Prace Mat.-Fiz. **47**, 41–48 (1949).

The author discusses the solution of the Dirac equation for an electron moving in the Coulomb field of a nucleus in momentum space. This equation is an integral equation

$$(*)\quad [E+\beta E_0+c(\alpha\cdot\mathbf{p})]v(\mathbf{p}) + (Ze^2/\pi h)\int v(\mathbf{p}')\,|\mathbf{p}-\mathbf{p}'|^{-2}d\mathbf{p}'=0,$$

where α and β are Dirac matrices. The dependence of v upon the angles of $\mathbf{p}$ may be directly expressed in terms of spherical harmonics. The integral equation (*) may then be reduced to two coupled integral equations in M and N:

$$\frac{1}{c}(E+E_0)M+pN+\frac{2Ze^2}{hc}\int_0^\infty \frac{p'}{p}MQ_{l+1}\frac{p^2+p'^2}{2pp'}dp'=0,$$

$$\frac{1}{c}(E-E_0)N+pM+\frac{2Ze^2}{hc}\int_0^\infty \frac{p'}{p}NQ_l\frac{p^2+p'^2}{2pp'}dp'=0,$$

where Q_{l+1} is the second solution of the Legendre differential equation. The author solves this pair of equations by reduction to a pair of differential equations. *H. Feshbach.*

Chandrasekhar, S., and Münch, G. On the integral equation governing the distribution of the true and the apparent rotational velocities of stars. Astrophys. J. **111**, 142–156 (1950).

The probability distribution $f(x)$ of the true rotational velocity x, and the distribution $g(y)$ of the apparent rotational velocity $y=x\sin i$ are connected by the relation $(*)$ $g(y)=y\int_y^\infty x^{-1}(x^2-y^2)^{-\frac{1}{2}}f(x)dx$. This can be reduced to Abel's integral equation and hence solved, but the solution contains differentiation and thus is awkward when $g(y)$ is an observed frequency function. A direct numerical solution of (*) meets with the same difficulty. The authors obtain the relation between corresponding moments of x and y so that the mean and mean square velocities can be determined from the observational data. For a more thorough investigation of the true distribution the authors advocate a comparison between the observed apparent velocity distribution and those derived from certain typical assumed forms of the true frequency function $f(x)$. They discuss briefly $f(x)=\delta(x-x_1)$, and $f(x)=x_1^{-1}$ for $0\leqq x\leqq x_1$ and $f(x)=0$ for $x>x_1$. The frequency function $f(x)=\pi^{-\frac{1}{2}}\{e^{-(x-x_1)^2}+e^{-(x+x_1)^2}\}$ is investigated in greater detail. For the corresponding

$$g(y;x_1)=\pi^{-\frac{1}{2}}y\int_y^\infty x^{-1}(x^2-y^2)^{-\frac{1}{2}}\{e^{-(x-x_1)^2}+e^{-(x+x_1)^2}\}dx$$

several expansions are obtained, and also some numerical results. [The mathematical reader should note that the authors do not always use standard notation. For instance $D_n(x_1)$ is not the familiar parabolic cylinder function but the integral (53). Curiously enough, this integral can be expressed in terms of parabolic cylinder functions. In the standard notation the integral (53) is

$$\pi^{-\frac{1}{2}}2^n e^{\frac{1}{2}x^2}\Gamma(\tfrac{1}{2}+n)\{D_{-2n-1}(x(-2)^{\frac{1}{2}})+D_{-2n-1}(-x(-2)^{\frac{1}{2}})\}.]$$

A. Erdélyi (Pasadena, Calif.).

Functional Analysis, Ergodic Theory

***Schwartz, L. Théorie des distributions. Tome I.** Actualités Sci. Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9. Hermann & Cie., Paris, 1950. 148 pp.

This is a generally clear, carefully organized, and detailed account of the basic aspects of the theory of "distributions" due to the author, and described by him in earlier publications [see especially Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 57–74 (1946); these Rev. **8**, 264]. This theory provides a convenient formalism for many common situations in theoretical and applied analysis, but its greatest significance may be in connection with partial differential equations, particularly those of hyperbolic type, where its adaptability to local problems gives it an advantage over Hilbert space (and other primarily global) techniques. On the other hand, the latter, when applicable, are considerably more powerful than any techniques as yet provided by distributions. A second volume in press is apparently to deal with real harmonic analysis and other applications of the general theory, and presumably will contain the bibliography needed for the identification of the references indicated in the present volume.

A "distribution" is a continuous linear functional on the space $\mathfrak{D}$ of functions on Euclidean space which are of class C^∞ and vanish outside compact sets, the continuity being relative to the topology of uniform convergence of the functions and all their derivatives, all the functions simultaneously vanishing outside the same fixed compact set. For example, any Lebesgue-Stieltjes measure defines a distribution via integration; and conversely every distribution which assigns nonnegative values to nonnegative functions arises in this way. Any function f integrable on compact sets likewise defines a distribution F by the equation $F(g)=\int fg$, and it is convenient to identify this distribution with the function. Every distribution T has "partial derivatives" defined by the equation $(\partial T/\partial x_k)(g)=-T(\partial g/\partial x_k)$; when T is a function this definition agrees with the usual one. All distributions are therefore indefinitely differentiable, and in particular, if the Dirac "function" at a point a is defined as the distribution arising from a measure of one concentrated at a, an explicit meaning has been attached to the derivatives of all orders of the Dirac function. The

"support" of a distribution is the complement of the unique maximal open set such that the distribution vanishes on all functions vanishing outside it.

The set $\mathfrak{D}'$ of all distributions is a topological linear space, the dual of the space $\mathfrak{D}$. The (dual) topology in $\mathfrak{D}'$ reduces to the following: A sequence T_n of distributions converges to a distribution T if $T_n(g) \to T(g)$ uniformly on bounded sets of elements g of $\mathfrak{D}$, such a set consisting of functions vanishing outside a fixed compact set which, together with their derivatives, are uniformly bounded ("bounded" as applied to subsets of $\mathfrak{D}$ has the usual meaning assigned in the theory of topological linear spaces). Then $\mathfrak{D}'$ is complete, $T(g)$ is jointly continuous on $\mathfrak{D}'$ and $\mathfrak{D}$, and most important of all, differentiation is a continuous linear operation on $\mathfrak{D}'$, so that convergent series in $\mathfrak{D}'$ can be differentiated term by term.

Every distribution coincides locally, i.e., inside a conditionally compact open set, with a derivative (of some finite order) of (the distribution arising from) a continuous function. A distribution whose support consists of a single point is a finite linear combination of the Dirac function and its derivatives. If S and T are distributions in spaces X and Y respectively, then there exists a unique "direct product" distribution $U = S \times T$ on $X \times Y$ such that $U(k) = S(g)T(h)$, if $k(x, y) = g(x)h(y)$; for measures, the result is the usual product measure. The map $(S, T) \to S \times T$ is jointly continuous on the spaces of distributions on X and Y to the space of distributions on $X \times Y$. This concept of direct product is to be used in the later treatment of convolutions of distributions. In general, distributions cannot be multiplied in the usual way, i.e., so as to generalize the multiplication of functions (formal generalizations of this type have actually been employed in physics) but an arbitrary distribution can always be multiplied by a function of class C^∞. A significant problem here is that of division of distributions by functions. This problem is not yet fully resolved, but it is always possible to divide by arbitrary monomials in the coordinates (in Euclidean space).

The book closes with a sketch of applications to partial differential equations. These are treated mainly in terms of their "elementary solutions," the elementary solution at a point a of a partial differential operator Γ being defined as a distribution T such that $\Gamma(T)$ is the Dirac function at a (this differs somewhat from the classical definition, but is formally the same as a definition employed in theoretical physics).

I. E. Segal (Chicago, Ill.).

Leray, Jean. Valeurs propres et vecteurs propres d'un endomorphisme complètement continu d'un espace vectoriel à voisinages convexes. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 177–186 (1950).

This paper shows how Riesz' theory of completely continuous operators may be extended from Banach spaces to (locally) convex topological linear spaces. Some results of the Banach space case, such as Schauder's theorem (that the transposed operator $T^*: \bar{X} \to \bar{X}$ is completely continuous if and only if $T: X \to X$ is) are not extended, however, since there is no obvious topology for the conjugate class $\bar{X}$ in the general case. The author assembles the theory in detail except that for some key theorems on invariance of the domain references are made to his earlier papers and to a book in preparation.

R. Arens (Los Angeles, Calif.).

Myers, S. B. Normed linear spaces of continuous functions. Bull. Amer. Math. Soc. **56**, 233–241 (1950).

Exposé systématique des récents travaux sur les relations entre un espace complètement régulier X, et l'espace de Banach $B(X)$ des fonctions numériques continues et bornées dans X (ou certains sous-espaces de $B(X)$). Un sous-espace vectoriel fermé B de $B(X)$ est dit complètement régulier sur X si, pour tout $x \in X$ et tout ensemble fermé $Y \subset X$ ne contenant pas x, il existe $b \in B$ telle que $b(x) = \|b\|$ et $\sup_{y \in Y} |b(y)| < \|b\|$. L'auteur donne des conditions nécessaires et suffisantes pour qu'un espace de Banach B soit isométrique à un sous-espace complètement régulier d'un $B(X)$, ainsi que les conditions supplémentaires moyennant lesquelles (1) X est compact, (2) (dans ce dernier cas) B est isométrique à $B(X)$ tout entier. La notion essentielle qui intervient dans ces conditions est celle de T-ensemble: ce sont les cônes engendrés par les parties convexes maximales de la sphère S: $\|x\| = 1$; il intervient aussi l'ensemble M des T-ensembles T tels que $T \cap S$ ne soit contenu que dans un seul hyperplan d'appui de S. L'auteur récapitule ensuite les propriétés topologiques simples de X dont on connaît la traduction en propriétés de $B(X)$. En particulier, il caractérise par une propriété de $B(X)$ les espaces péaniens X; cette caractérisation est déduite du théorème suivant. Si X_1 et X_2 sont deux espaces compacts, pour qu'il existe une application continue de X_1 sur X_2, il faut et il suffit qu'il existe une isométrie de $B(X_2)$ sur un sous-espace B de $B(X_1)$, tel que l'intersection de tout T-ensemble de $B(X_1)$ avec B soit un T-ensemble de B. *J. Dieudonné.*

Mandelbrojt, Szolem. Un théorème de fermeture. C. R. Acad. Sci. Paris **231**, 16–18 (1950).

If $f(x) \in L(-\infty, \infty)$, let $\pi(f)$ be the closed linear subspace spanned by $f(x)$ and its translations $f(x+\xi)$. If $f(x)$ has derivatives of all orders and $f^{(n)}(x) \in L(-\infty, \infty)$ for all n, let $\omega(f)$ be the closed linear subspace spanned by $f(x)$ and its derivatives. If $\|f^{(n)}\| < M_n$, $f(x)$ is said to belong to the class $L\{M_n\}$. Here $\omega(f) \subset \pi(f)$ and $\omega(f') \subset \pi(f')$, but ordinarily $\pi(f') \subset \omega(f')$ does not hold. A necessary and sufficient condition that $\pi(f') = \omega(f')$ for each $f \in L\{M_n\}$ is that the class $L\{M_n\}$ is quasi-analytic. A proof for the sufficiency is outlined; it depends upon the fact that an essentially bounded function $h(x)$ normalized by $\int_{-\infty}^{\infty} f'(x+a)h(x)dx = 1$ cannot be orthogonal to $f^{(n)}(x)$ for all $n \geqq 2$ whence it follows that the translation $f'(x+a)$ can be approximated in the mean by linear combinations of the higher derivatives. The author states that conditions can be found on the constants $\{M_n\}$ and the integers $\{\nu_n\}$ so that the derivatives of $f(x)$ of order ν_n can always be used to approximate $f'(x+a)$ whenever $f(x) \in L\{M_n\}$. No conditions are given, however.

E. Hille (New Haven, Conn.).

Birman, M. Š. Some estimates for the method of steepest descent. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 152–155 (1950). (Russian)

Let A be a symmetric linear operator in Hilbert space which satisfies $m(x, x) \leqq (Ax, x) \leqq M(x, x)$ where $0 < m < M$. Finding the solution x^* of the equation $Ax - \varphi = 0$ is equivalent to finding the minimum of $H(x) = (Ax, x) - 2(x, \varphi)$. The method of steepest descent is an iterative process for finding this minimum which has been investigated by Kantorovič [e.g., same journal (N.S.) **3**, no. 6(28), 89–185 (1948); these Rev. **10**, 380]. In the present paper the iteration scheme is given by $x_n = x_{n-1} + \sum_{k=0}^{r-1} \alpha_k^{(n)} A^k z_n$, where x_0 is arbitrary, $z_n = Ax_{n-1} - \varphi$, and the numbers $\alpha_k^{(n)}$ are deter-

mined so that $H(x_n)$ is a minimum. The author proves the following inequalities:

$$H(x_n)-H(x^*)\leqq L_p^{2n}[H(x_0)-H(x^*)],$$
$$\|x_n-x^*\|\leqq m^{-1}L_p^n[H(x_0)-H(x^*)]^{\frac{1}{2}},$$

where

$$L_p^{-1}=\tfrac{1}{2}\left\{\frac{M+m}{M-m}-\left[\left(\frac{M+m}{M-m}\right)^2-1\right]^{\frac{1}{2}}\right\}^p+\tfrac{1}{2}\left\{\frac{M+m}{M-m}+\left[\left(\frac{M+m}{M-m}\right)^2-1\right]^{\frac{1}{2}}\right\}^p.$$

The estimates cannot be improved for the class of all operators A of the type above. The paper provides an answer to a problem of Kantorovič [loc. cit., p. 145, remark 3], who treated the case $p=1$, but mentions the possibility of obtaining more rapid convergence with $p>1$.

J. V. Wehausen (Providence, R. I.).

Iohvidov, I. S. On the spectra of Hermitian and unitary operators in a space with indefinite metric. Doklady Akad. Nauk SSSR (N.S.) 71, 225–228 (1950). (Russian)

L. Pontrjagin [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 8, 243–280 (1944); these Rev. 6, 273] studied the separable infinite Hilbert space H_k with the ordinary Hermitian metric $|x|=(\sum_{n=1}^{\infty}|x_n|^2)^{\frac{1}{2}}$ but with the modified notion of "scalar" product

$$[x,y]=-\sum_{i=1}^{k}x_iy_i+\sum_{i=k+1}^{\infty}x_iy_i, \qquad k\geqq 1. \tag{1}$$

The "direct" sum of subspaces and the "adjoint" A^* of an operator A are defined in the usual way in terms of (1). An operator A is said to be "Hermitian" ("self-adjoint") if $A=A^*$; an operator U is said to be "unitary" if it is a linear one-to-one mapping of H_k onto itself, preserving $[x,y]$. A vector x is said to be "positive" ("negative") if $[x,y]>0$ (<0). The author [in a paper unavailable outside the Soviet Union] has transferred Pontrjagin's results concerning "Hermitian" operators A to the corresponding "unitary" operators U, by the following Cayley relations:

$$A=\mu i(\alpha I+U)(\alpha I-U)^{-1}, \quad U=\alpha(\mu iI-A)(\mu iI+A)^{-1},$$

where $|\alpha|=1$, α is not a characteristic value of U, and $-\mu i$ is not a characteristic value of A, $\mu>0$. The author's main result in the paper under consideration is that the whole spectrum of any "unitary" U in H_k lies on the unit circle, except for at most k pairs of eigenvalues which are symmetric with respect to this circle. The case $k=1$ was established by Kreĭn and Rutman [Uspehi Matem. Nauk (N.S.) 3, no. 1(23), 3–95 (1948); these Rev. 10, 256]. We give some explanatory details: Since U is "unitary," there exist two k-dimensional invariant subspaces T, T' (in general with $TT'\neq(0)$), composed of non-"positive" vectors; the characteristic values of U, restricted to T and T', have modulus $\geqq 1$, $\leqq 1$ respectively; T, T' contain all the characteristic manifolds T_λ for $|\lambda|>1$, $|\lambda|<1$ respectively; to every characteristic value λ with $|\lambda|\neq 1$ there corresponds another, $1/\lambda$, with the same composition and same order of elementary divisors. Let L be the direct sum of all T_λ, $\lambda\neq 1$. On L the form $[x,x]$ is not degenerate (i.e., the only vector in L "orthogonal" to L is 0). The subspace $H_{k'}$, defined by the "direct" sum $H_k=L+H_{k'}$, is an invariant subspace of U and $k'\leqq k$, and U has within $H_{k'}$ all its eigenvalues on the unit circle. The author, following the terminology used in the cited work of Kreĭn and Rutman, calls L the hyperbolic part. Its dimension $2l$ is such that $0\leqq l\leqq k$. If $l<k$, then $k'=k-l$, and $H_{k'}$ may contain "negative" eigenvalues of U. Let M be the subspace of $H_{k'}$, "negative" invariant with respect to U, and with maximal dimension. Then M is called the elliptic part, and its dimension is less than or equal to k'; L as well as M may be (0). The subspace $H_{k''}$ defined by the "direct" sum $H_{k'}=M+H_{k''}$ is termed the parabolic part. We have $k''\leqq k'$. For the case $k''=0$, $H_{k''}$ is an ordinary Hilbert space on which U is an ordinary unitary operator. Details are given of the behaviour of U in the case $k''>0$, to which the work is especially devoted. They are obtained by the examination of the set $\mathfrak{M}$ of all vectors in H_k "orthogonal" to T, and by considering the common part of $\mathfrak{M}$ and T. In the case where U is simple (i.e., has no invariant "positive" subspaces), some results are stated. There are applications to totally continuous "Hermitian" operators, and also to weighted integral equations $\varphi(x)=\lambda\int_a^b k(x,s)\varphi(s)d\omega(s)$, where $\omega(s)=\sigma(s)-\tau(s)$ with bounded and nondiminishing σ and τ and where $\tau(s)$ is a step-function admitting k values. In regard to this the author cites M. G. Kreĭn [memorial volume dedicated to D. A. Grave, Moscow, 1940, pp. 88–103; these Rev. 2, 312]. There are no proofs. *O. M. Nikodým* (Gambier, Ohio).

Kakutani, Shizuo. Determination of the spectrum of the flow of Brownian motion. Proc. Nat. Acad. Sci. U. S. A. 36, 319–323 (1950).

A Brownian motion defines a group of measure-preserving transformations, the transformations generated by such translations in time. Let L_2 be the L_2 space of functions defined on the measure space of the Brownian motion. The group of measure-preserving transformations generates a group of unitary transformations acting on L_2. A proof is sketched that L_2 can be decomposed into a direct sum of countably many mutually orthogonal closed linear manifolds $\mathfrak{M}_0, \mathfrak{M}_1, \cdots$ in such a way that $\mathfrak{M}_n$ is invariant under the unitary group just described, and that this group on $\mathfrak{M}_n$ is spectrally isomorphic with the unitary group defined by $V_t^{(n)}f(s_1,\cdots,s_n)=f(s_1+t,\cdots,s_n+t)$, where f is an element of the L_2 space of functions whose squares are Lebesgue integrable in n dimensions. *J. L. Doob* (Urbana, Ill.).

Fomin, S. V. On measures invariant under certain groups of transformations. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 261–274 (1950). (Russian)

The main purpose of this paper is to generalize the results of Kryloff and Bogoliouboff [Ann. of Math. (2) 38, 65–113 (1937)] to not necessarily one-parametric groups. Suppose that R is a compactum, S is a topological group acting continuously on R, and G is the multiplicative group of all continuous functions on R whose values are complex numbers of modulus 1. If μ is a probability measure in R, invariant under S, and if, for every f in $L_2(\mu)$, $A_{g,s}f(x)=g(x)f(sx)$ whenever $g\in G$ and $s\in S$, then (theorem 1) $(g,s)\to A_{g,s}$ is a unitary representation of the crossed product of G and S; this representation is irreducible (theorem 2) if and only if μ is ergodic. A probability measure invariant under S exists whenever S is compact (theorem 3), or, more generally, S has a closed, invariant subgroup T which leaves invariant at least one probability measure and for which S/T is compact (theorem 4), or S is Abelian (theorem 5).

Kryloff and Bogoliouboff proved that if S is the real line, then to each ergodic probability measure μ invariant under S there corresponds a (strictly) invariant set E_μ such that $\mu(E_\mu)=1$ and such that $\mu'(E_\mu)=0$ for every ergodic probability measure μ' ($\neq\mu$) invariant under S. The author presents a (hitherto unpublished) example of Kolmogoroff's which shows that this result is not true for every group S,

and he asks whether or not the result remains true if the requirement of strict invariance is dropped. As a first step toward the answer he proves (theorem 6) that any two ergodic probability measures invariant under S are mutually singular.

The paper concludes with two comments on convex sets. A convex subset N of a convex set M in the conjugate space of a separable Banach space is called a regular part of M if every extreme point of N is an extreme point of M. If, for each point p of M, $M(p)$ is the intersection of M and the family of all supporting planes containing p, then (theorem 7) $M(p)$ is always a regular part of M. The last comment (theorem 8) asserts that if M is the set of all probability measures in R, then the natural correspondence between R and the set of all extreme points of M is a homeomorphism (with respect to the weak topology of M).

P. R. Halmos (Chicago, Ill.).

Fréchet, Maurice. Les transformations asymptotiquement presque périodiques discontinues et le lemme ergodique. I. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 61–68 (1950).

A function $F(t)$ defined for real numbers $t \geqq \alpha$ and belonging to a class $\mathfrak{F}$, to which a limit topology has been assigned, is asymptotically almost periodic to the right provided any sequence of real numbers $\{h_n\}$ tending to $+\infty$ contains a subsequence $\{k_n\}$ such that the sequence $\{F(t+k_n)\}$ converges in $\mathfrak{F}$. The author makes various remarks about this property and applies them to the case of a continuous one-parameter semiflow on a compact space. Under strong hypotheses he drives a result concerning means at infinity and states that he has proved more than the ergodic lemma. It is difficult to see the significance of these results as far as ergodic theory is concerned. *G. A. Hedlund.*

Kaplan, Wilfred. Dynamical systems with indeterminacy. Amer. J. Math. 72, 573–594 (1950).

A classical dynamical system (system of ordinary differential equations) is deterministic in the sense that specified initial conditions determine the motion precisely. In this paper the author initiates a study of dynamical systems (flows in compact n-dimensional Riemannian manifolds M) in which the velocity vector of a motion is permitted to differ from a prescribed vector by a specified amount ϵ, such a solution being called an ϵ-solution. If there is an ϵ-solution from x to y with increasing t it is said that $x<y$. If $x<y<x$ or if $x=y$, x and y are said to be equivalent and this equivalence relation partitions M into states. If a state consists of one element it is transitory, otherwise it is stationary. The state containing x is denoted by $\pi(x)$ and the set of all y such that $y<x$ is denoted by $\alpha(x)$. A stationary state $\pi(x)$ is ω-stable (α-stable) if $x<y$ ($x>y$) implies $y \varepsilon \pi(x)$. It is easy to show that if M is compact, there are only a finite number of ω-stable states $\pi(z_1), \cdots, \pi(z_N)$ and $x \varepsilon M$ implies $x<z_j$ for some j, with similar results for α-stability. The sets $\alpha(z_j)$ are open and cover M. The nerve of this covering is a complex Φ which, except for certain critical choices of ϵ, can be computed by a finite process. The situation is illustrated by use of the van der Pol equation to define the prescribed vector field. *G. A. Hedlund.*

Nemyckiĭ, V. V. Generalizations of the theory of dynamical systems. Uspehi Matem. Nauk (N.S.) 5, no. 3(37), 47–59 (1950). (Russian)

This paper is an expository account of known results. In line 9 of page 57 the author translates "relatively dense" into the Russian equivalent of "everywhere dense."

W. H. Gottschalk (Philadelphia, Pa.).

Theory of Probability

Rényi, Alfréd. On the algebra of distributions. Publ. Math. Debrecen 1, 135–149 (1950).

Let G be a distribution function, and let $\lambda_1, \cdots, \lambda_n$ be real numbers different from zero. If there exists a distribution function F, and mutually independent random variables $\xi_1, \cdots, \xi_n$, each with distribution function F, such that the sum $\sum_1^n \lambda_j \xi_j$ has distribution function G, then the author writes $F=G/(\lambda_1, \cdots, \lambda_n)$. The following are some of the results obtained. (1) If the G distribution is bounded, then F is uniquely determined by the relation $F=G/(\lambda_1, \cdots, \lambda_n)$, if there is one solution, if and only if $\sum_1^n \lambda_i^s \neq 0$. If, in addition, it is supposed that $\sum_1^n \lambda_i^s \neq 0$, $s=1, 2, \cdots$, the characteristic function of F is found explicitly in terms of that of G. (2) In the sufficiency half of (1) the hypothesis that the G distribution is bounded can be replaced either by the hypothesis that the characteristic function of G is analytic in an interval containing the origin, and that $\sum_1^n \lambda_i^s \neq 0$, $s=1, 2, \cdots$, or by the hypothesis that $n=2$ and that the characteristic function of G does not vanish. Let $\xi_1, \cdots, \xi_n$ be mutually independent random variables with distribution functions $F_1, \cdots, F_n$ and let G_j be the distribution function of $\sum_{i=1}^n c_{ji}\xi_i$, $j \leqq n$. The author then writes $(F_1, \cdots, F_n)=(G_1, \cdots, G_n)/\Delta$, where Δ is the matrix (c_{ij}). (3) It is proved that if the characteristic function of G_1 is an entire function of order not greater than 2, then the F's are uniquely determined by the relation

$$(F_1, \cdots, F_n)=(G_1, \cdots, G_n)/\Delta,$$

if there is a solution, if and only if the determinants $|c_{ji}|$ and $|c_{ji}^2|$ do not vanish.

J. L. Doob (Urbana, Ill.).

Theil, H. A note on the inequality of Camp and Meidell. Statistica, Rijswijk 3, 201–208 (1949). (Dutch. English summary)

A supposedly new and simpler proof of the generalized Gauss inequality connecting tail areas with the absolute moments of probability laws with convex distribution functions is, in fact, that of von Mises [Wahrscheinlichkeitsrechnung . . . , Deuticke, Leipzig-Wien, 1931].

H. L. Seal (New York, N. Y.).

Sapogov, N. A. On a multidimensional limit theorem of the theory of probability. Uspehi Matem. Nauk (N.S.) 5, no. 3(37), 137–151 (1950). (Russian)

Detailed exposition of a simple way of reducing the multidimensional central limit theorem to the one-dimensional case. Use is made of a theorem of Cramér and Wold [J. London Math. Soc. 11, 290–294 (1936)]. *W. Feller.*

Loève, Michel. Lois pondérées et le problème limite central. C. R. Acad. Sci. Paris 231, 26–28 (1950).

The author states the heuristic principle that if in studying the limit laws of sums of mutually independent individually negligible random variables, the hypothesis of independence is dropped, the limit laws become weighted averages of the limit laws in the independence case. Extensions of earlier results [same C. R. 230, 52–53 (1950); these Rev. 11, 375] generalizing the ergodic theorem are stated. [The author wishes to call attention to the following errata: In line 11 read $\{\mathfrak{L}(\sum_k X_{nk})\}$ instead of $\{\mathfrak{L}(X_{nk})\}$. In line 14 read "Sous" instead of "Sans."] *J. L. Doob.*

Moran, P. A. P. Notes on continuous stochastic phenomena. Biometrika **37**, 17–23 (1950).

(1) The author considers a stochastic process

$$\{x(t), -\infty < t < \infty\}$$

which is stationary in the wide sense, and defines

$$X(p) = \int_p^{p+1} x(t)dt,$$

where p is an integer. The $X(p)$ process is then a discrete parameter process, stationary in the wide sense, and the correlation function of this process is evaluated under the assumption that the $x(t)$ process has correlation function $\exp(-\lambda|t|)$. The parameter t is then replaced by a number pair. To obtain a process whose parameter domain is the plane, it is supposed that

$$E\{x(s+s_1, t+t_1)\overline{x(s, t)}\} = \exp(-\lambda|s_1| - \mu|t_1|).$$

An $X(p, q)$ process is defined in the natural generalization of the averaging procedure used above into two dimensions, and the two-dimensional correlation function of the $X(p, q)$ process is derived. (2) Let x_{ij}, $i \leqq m$, $j \leqq n$, be mn mutually independent random variables with a common distribution. A natural definition is made of a sample correlation coefficient r_{11} between x's which are nearest neighbors. The first moment of r_{11} is found, and also the second moment under the assumption that the x's are Gaussian. *J. L. Doob.*

Bartlett, M. S. Periodogram analysis and continuous spectra. Biometrika **37**, 1–16 (1950).

The author discusses the inadequacy of the classical procedure of harmonic analysis when applied to sample sequences from a stochastic process $\{x_n, -\infty < n < \infty\}$ which is stationary in the wide sense and has a continuous spectral distribution. He proposes another procedure in which, if sample values of $x_1, \cdots, x_{mn}$ are available, the usual procedure is applied to sections of length n and averaged. This procedure is tested on several empirical examples.

J. L. Doob (Urbana, Ill.).

Hamming, R. W. Error detecting and error correcting codes. Bell System Tech. J. **29**, 147–160 (1950).

In this paper the author considers how best to introduce redundancies into a code in such a way that an error in the transmission or reception of the information contained in the code can be detected or corrected. The paper consists of two major parts. In part I, the author constructs special codes which can either detect a single error, correct a single error, or correct a single error and detect a double error. In part II, he discusses the general theory and shows that under suitable assumptions these codes are optimal. The method of attack in part II is geometrical.

H. H. Goldstine (Princeton, N. J.).

Shannon, Claude E. Memory requirements in a telephone exchange. Bell System Tech. J. **29**, 343–349 (1950).

Spring, W. Analytische Betrachtungen zur Änderung des Rechnungszinsfusses und der Sterbetafel bei Versicherungswerten. Mitt. Verein. Schweiz. Versich.-Math. **50**, 111–132 (1950).

H. Hadwiger [same Mitt. **45**, 31–35 (1945); these Rev. **7**, 122] gave an exact solution for the problem of interest in actuarial mathematics. His solution was given in such a form that it applied to a wide variety of actuarial functions. The present author reproduces Hadwiger's proof and discusses various actuarial functions in detail. He shows that a similar treatment can be applied also to changes in mortality. This is almost trivial since only the sum of the force of mortality and of the force of interest occur in the differential equation for the actuarial functions under consideration.

E. Lukacs (Washington, D. C.).

Mathematical Statistics

***Yule, G. Udny, and Kendall, M. G. An Introduction to the Theory of Statistics.** 14th ed. Hafner Publishing Co., New York, N. Y., 1950. xxiv+701 pp.

This is a new revision of the well-known textbook intended for those who possess only a limited knowledge of mathematics. The first edition appeared in 1911 [Griffin, London], a thorough revision was made in 1937 [Griffin, London]. The first half of the book deals with descriptive (nonsampling) statistics (theory of attributes, frequency distributions and their characteristics, correlation and regression, curve fitting). The bulk of the second half is concerned with sampling theory (large and small samples, chi-square, analysis of variance). The last chapters discuss interpolation and graduation, index numbers, and time series. Compared with the previous edition the principal changes are: expanded treatment of small-sample theory and analysis of variance; new chapters on problems of practical sampling, index numbers and time-series; condensation of the treatment of the theory of attributes; omission of the list of references [an extensive bibliography has appeared in Kendall, Advanced Theory of Statistics, vol. 2, Griffin, London, 1946; these Rev. **8**, 473]. The present edition covers probably a greater variety of subjects than any other comparable book. It still bears the imprint of the Pearsonian school and Yule's work, but many more recent developments have been taken into account in the revisions, made chiefly by Kendall. Unfortunately the "classical" failure to distinguish by a clear notation between sample frequencies and expected frequencies, sample moments and population moments still persists in the chapters on large sample theory (although the distinction is made in the treatment of "small" samples). The detailed discussion of the various statistical techniques particularly emphasises the underlying assumptions and the interpretation of results and is accompanied by numerous illustrations and exercises. *W. Hoeffding.*

***v. Baranow, L. Grundbegriffe moderner statistischer Methodik. Erster Teil. Merkmalsverteilungen.** S. Hirzel Verlag, Stuttgart, 1950. viii+112 pp. DM 6.00.

***v. Baranow, L. Grundbegriffe moderner statistischer Methodik. Zweiter Teil. Zeitliche und kausale Zusammenhänge.** S. Hirzel Verlag, Stuttgart, 1950. vii+111 pp. DM 6.50.

Proposals for the standardization of symbols in mathematical statistics and biometrics. Statistica, Rijswijk **4**, 80–85 (1950). (Introduction in Dutch, main text in English)

Aitken, A. C. On the statistical independence of quadratic forms in normal variates. Biometrika **37**, 93–96 (1950).

Let x be a vector having as components n uncorrelated variates, each with mean zero and variance unity. The author proves rigorously Craig's theorem: A necessary and sufficient condition for the quadratic forms $x'Ax$ and $x'Bx$

to be independent is that $AB=0$. Use is made of the identity (in α and β), $|I-\alpha A-\beta B|=|I-\alpha A||I-\beta B|$, which is necessary and sufficient for independence. Some extensions of the theorem are given. *T. W. Anderson.*

Birnbaum, Z. W., Paulson, E., and Andrews, F. C. On the effect of selection performed on some coordinates of a multi-dimensional population. Psychometrika 15, 191–204 (1950).

Let the vectors X and Y have the joint density

$$f(X, Y)=g(X)h(Y|X),$$

where $g(X)$ is the marginal density of X and $h(Y|X)$ is the marginal density of Y given X. "Selection" means changing the marginal density of X to $g^*(X)$ giving a new joint density $f^*(X, Y)=g^*(X)h(Y|X)$. A theorem of Lawley [Proc. Roy. Soc. Edinburgh. Sect. A. 62, 28–30 (1943); these Rev. 5, 127] is used to study the question of what knowledge beyond the first and second moments of $f^*(X, Y)$ is sufficient to determine the first and second moments of $f(X, Y)$. Some asymptotic variances are obtained and some examples are given. *T. W. Anderson* (New York, N. Y.).

Hayashi, Chikio. Fragments of a new test formula of normality. Ann. Inst. Statist. Math., Tokyo 1, 125–130 (1950).

Let $X_1, \cdots, X_{2n+1}, \cdots, X_N$ be independent normal random variables with zero mean and common variance σ^2. Let $\tilde{x}$ be the median of $X_1, \cdots, X_{2n+1}$ and let s^2 be the usual estimate of σ^2 based on $X_{2n+2}, \cdots, X_N$. The author obtains an expression for the distribution of $\tilde{x}/s$ and discusses various approximations to this distribution. Applications to testing a sample for normality are promised for a later paper. *E. L. Lehmann* (New York, N. Y.).

Kaloujnine, L. Quelques idées au sujet du mémoire de M. G. Neymann "L'estimation statistique traitée comme un problème classique de probabilité." Publ. Math. Debrecen 1, 101–103 (1949).

Expository paper. *J. Wolfowitz* (New York, N. Y.).

Midzuno, Hiroshi. An outline of the theory of sampling systems. Ann. Inst. Statist. Math., Tokyo 1, 149–156 (1950).

The author approaches the problem of estimating a parameter of a finite universe when the probability of sampling elements of the universe is not equally likely. An unbiased estimate is constructed if the actual sampling probabilities are known and if an unbiased estimate of the parameter is known under random sampling. If the sampling probabilities are at the statistician's disposal then it is possible to consider an optimum sampling procedure. The results in this direction are based upon a vague approximation (without any appraisal of the error term) and an incorrect maximization procedure. *D. G. Chapman* (Seattle, Wash.).

Bhattacharyya, A. Unbiased statistics with minimum variance. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 69–77 (1950).

Let X be a random vector with probability density $F(x; \theta)$. It is desired to determine a minimum variance unbiased estimate of $\tau(\theta)$. Let $\Phi_j(X, \theta)$ be functions such that the expectation of $t(X)\Phi_j(X, \theta)$ is the same for all unbiased estimates $t(X)$ of $\tau(\theta)$. Such functions can be obtained, for example, by differentiating or integrating the equation of unbiasedness. The author proves that if there exist real constants λ such that $T=\lambda_0+F^{-1}\sum\lambda_j\Phi_j$ is independent of θ and unbiased for estimating $\tau(\theta)$, then T has minimum variance among all unbiased estimates of $\tau(\theta)$. Some examples are given. *E. L. Lehmann* (New York, N. Y.).

Olekiewicz, M. On the efficiency of biased estimates. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 3, 103–140 (1949). (English. Polish summary)

Let $\hat{\alpha}$ be an unbiased estimate of a population parameter α. Then for any $a\neq 0$, $\alpha^*=(1+a)\hat{\alpha}$ is a biased estimate with linear bias $a\alpha$. The author shows that the value of a which minimizes $E(\alpha^*-\alpha)^2$ is given by $a_0=-D^2(\hat{\alpha})/[\alpha^2+D^2(\hat{\alpha})]$ where $D^2(\hat{\alpha})$ is the sampling variance of $\hat{\alpha}$. It follows that if only linearly biased estimates are considered, the Cramér-Rao lower limit for $E(\alpha^*-\alpha)^2$ may be replaced by $\alpha^2/[1+\alpha^2 n\int_{-\infty}^{\infty}(\partial \log f(x)/\partial\alpha)^2 f(x)dx]$ and a corresponding expression in the discrete case. The results are applied to the variance σ^2 of a normal distribution. In this particular case, the author also investigates biased estimates which minimize $E|\sigma^{2*}-\sigma^2|$. *G. E. Noether* (New York, N. Y.).

Hodges, J. L., Jr., and Lehmann, E. L. Some problems in minimax point estimation. Ann. Math. Statistics 21, 182–197 (1950).

It is shown that if the loss function is convex, and if only point estimates with finite expectation are considered, the class of nonrandomized point estimates is essentially complete, i.e., for any randomized estimate τ there exists a nonrandomized estimate t such that the risk function of t nowhere exceeds the risk function of τ. A theorem of Blackwell on the uniform improvement of the variance of an unbiassed estimate by taking conditional expectation with respect to a sufficient statistic [same Ann. 18, 105–110 (1947); these Rev. 8, 478] and its extension by Barankin [same Ann. 21, 280–284 (1950); these Rev. 11, 732] are shown to be simple consequences of the above result. Assuming the loss function to be the squared error, minimax estimates are found explicitly for several specific parametric and nonparametric problems. Among the results obtained are the following. (1) In the case of a binomial variable, the minimax estimate of the probability p of success in a single trial, based on a sample of n independent trials, is given by $(n^{\frac{1}{2}}+1)^{-1}(\bar{p}n^{\frac{1}{2}}+\frac{1}{2})$, where $\bar{p}$ is the observed proportion of successes. (2) If $X_1, \cdots, X_n$ is a sample from an unknown univariate distribution over $[0, 1]$, then the minimax estimate of $E(X_i)$ is given by the same formula as in the binomial case, except that $\bar{p}$ is replaced by $\bar{X}=(X_1+\cdots+X_n)/n$. Also, some prediction problems are solved using the minimax principle. *A. Wald* (New York, N. Y.).

Wolfowitz, J. Minimax estimates of the mean of a normal distribution with known variance. Ann. Math. Statistics 21, 218–230 (1950).

Let $x_1, x_2, \cdots$ be independent normal variables with variance 1 and unknown mean ξ, and suppose that the cost of n observations $x_1, \cdots, x_n$ is cn. The author considers the following four problems: (1) estimation of ξ by an interval I of fixed length, with loss equal to $P\{\xi\bar{\in}I\}+cn$, (2) specifying an upper bound L for ξ with loss equal to $P\{\xi\geqq L\}+cn+d(L-\xi)^2$, (3) estimation of ξ by an interval I of varying length, with loss equal to $P\{\xi\bar{\in}I\}+cn+dL^2$, where L is the length of I, and (4) estimation of ξ by a number t, with loss equal to $cn+R(|\xi-t|)$, where R is any nondecreasing function of $|\xi-t|$. Among all sequential procedures, it is shown that the minimax procedure for each

of these problems is a sample of some fixed size N, followed by a classical estimation procedure for the particular problem. The method in each case is to find the Bayes solution when ξ is $N(0, \sigma^2)$ and allow σ^2 to become infinite.

D. Blackwell (Stanford University, Calif.).

Das, A. C. Two dimensional systematic sampling and the associated stratified and random sampling. Sankhyā 10, 95–108 (1950).

Results of Cochran [Ann. Math. Statistics 17, 164–177 (1946); these Rev. 8, 43] are generalized to two-dimensional sampling. The author gives the expected values of the sample variances of the mean under random, stratified, and systematic sampling. He then investigates sufficient conditions for stratified sampling to be more efficient than random sampling, and for systematic sampling to be more efficient than random sampling. *J. Wolfowitz* (New York, N. Y.).

Hitchcock, H. P. The estimation of the probable error from successive and independent variances. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 193, 23 pp. (1940).

Given x_i, $i=1, 2, \cdots, n$, distributed normally with variance σ^2, define $s_n^2=\sum_{i=1}^{n}(x_i-n^{-1}\sum_{j=1}^{n}x_j)^2/n$. Further let $s^2_{m,n}$, called the average successive variance of n measurements in groups of m, be defined by

$$(n-m+1)s^2_{m,n}=\frac{m-1}{m^2}\sum_{i=1}^{n}c_ix_i^2-\frac{2}{m^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}c_{ij}x_ix_j.$$

The author tabulates c_i, and c_{ij}, for $m=2(1)5$, and n, finds the variance of $s^2_{m,n}$, and the ratio of $s^2_{m,n}$ to the variance of $s^2_{m,n}$. The results are extended to the case of independent variances. This is followed by a series of conclusions in which the author recommends various methods of estimating σ^2 under different circumstances. The efficiencies of these methods are not discussed. *L. A. Aroian.*

Bellinson, H. R. The distribution of the estimate of standard deviation obtained by the method of successive differences. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 200, 20 pp. (1940).

The author finds the third and fourth central moments of $\delta_n^2=\sum_{i=1}^{n-1}(x_i-x_{i+1})^2/(n-1)$, where $x_1, \cdots, x_n$, is a sample from a normal probability function. In conjunction with the mean and variance of δ_n^2, the distribution of δ_n^2 is fitted by a Pearson type VI curve. This differs from the results of von Neumann, Kent, Bellinson, and Hart [Ann. Math. Statistics 12, 153–162 (1941); these Rev. 3, 7], where the distribution is fitted by taking the origin at zero and using the first three moments of δ_n^2. *L. A. Aroian.*

Matusita, Kameo. A remark to the Wald's theory of statistical inference. Ann. Inst. Statist. Math., Tokyo 1, 141–148 (1950).

The author gives general conditions under which the main results of a paper by Wald [Ann. of Math. (2) 46, 265–280 (1945); these Rev. 7, 21] are obtainable. The sample space and the space of distribution functions are to be topological spaces satisfying certain conditions. Certain continuity restrictions are imposed on the weight function.

J. Wolfowitz (New York, N. Y.).

Stevens, W. L. Fiducial limits of the parameter of a discontinuous distribution. Biometrika 37, 117–129 (1950).

The author shows that there are no nonrandomized confidence intervals for binomial p with confidence coefficient exactly equal to α, $0<\alpha<1$, for all p (this was proved earlier by Neyman [Ann. Math. Statistics 6, 111–116 (1935)]), and he points out that exact confidence intervals can be obtained if one adds to the binomial variable a continuous variable with known distribution. It should be mentioned that the kind of confidence intervals one obtains in this manner are just what one would get by specializing Wald's randomized decision functions to the problem of confidence intervals. *E. L. Lehmann* (New York, N. Y.).

Roy, S. N. Univariate and multivariate analysis as problems in testing of composite hypotheses. I. Sankhyā 10, 29–80 (1950).

Some general remarks concerning various approaches to the problem of hypothesis testing are followed by a discussion of the problem of obtaining optimum regions among a certain class of similar regions (those having the well-known structure with respect to a set of sufficient statistics). Applications are given to the general linear univariate hypothesis and to the hypothesis of equality of two variances. Possibly this work was done at a time when the author did not have ready access to the literature. While his treatment differs in some respects from the customary one, the main results are essentially contained in papers by P. L. Hsu, Wald, Scheffé, and the reviewer. [For references see the bibliography in Lehmann, Ann. Math. Statistics 18, 473–494 (1947); these Rev. 9, 454.] An exception is the treatment of linear hypotheses as multiple classification problems. Here the author shows how to minimize the total risk under the assumption of given a priori probabilities.

E. L. Lehmann (New York, N. Y.).

Narain, R. D. On the completely unbiassed character of tests of independence in multivariate normal systems. Ann. Math. Statistics 21, 293–298 (1950).

Consider n observations of q sets of random variates with a multivariate normal distribution. A test for the independence of the q sets of variates consists of using Wilks' statistic. The unbiased character of this test is established by successively establishing the unbiased character of the likelihood ratio tests for (1) the mean of a normal population equal to zero, (2) the partial regression coefficients of a variate x_p on $x_m, x_{m+1}, \cdots, x_{p-1}$ equal to zero (the test for the significance of the multiple correlation coefficient is a special case) and (3) the independence of two sets of variates. The arguments involve showing unbiasedness with respect to the partial regression coefficients of the last variate or set of variates first, making use of independence resulting from setting the above coefficients equal to zero and proceeding to the next to last set of variates. In no case is the distribution of the test statistic required.

H. Chernoff (Urbana, Ill.).

Hemelrijk, J. A family of parameterfree tests for symmetry with respect to a given point. I. Nederl. Akad. Wetensch., Proc. 53, 945–955 = Indagationes Math. 12, 340–350 (1950).

Let $z_1, z_2, \cdots, z_n$ be independent observations on n random variables; it is desired to test the null hypothesis that the variables are symmetrically distributed about a common median, e.g., zero; their distributions may all be different and need not be continuous. Suppose p of the observations are positive, q are negative, and m are zero. Divide the $p+q$ nonzero z's into two sets of as nearly equal size as possible so that every absolute value in the larger set is greater than every absolute value in the smaller set.

(The two sets may be of equal size.) Let u of the positive observations be in the larger set (which contains r observations) and $p-u$ in the smaller set. The joint density for u and $p-u$ given m and r under the null hypothesis is $\binom{r}{u}\binom{p+q-r}{p-u}\Big/2^{p+q}$. Various possibilities for critical regions in the $(u, p-u)$ space are discussed.

A. M. Mood (Santa Monica, Calif.).

David, F. N. Two combinatorial tests of whether a sample has come from a given population. Biometrika **37**, 97–110 (1950).

The following test is proposed for testing the hypothesis H_0 that a random sample has come from a given continuous population. Divide the range of an individual observation into k disjoint sets of equal probability under H_0 and count the number t of sets which contain no observations. Large values of t are significant. The choice $k=N$ is recommended. A modification of the test is proposed for the case when the class C of alternatives is restricted. The range is divided into $2N$ sets, N of them are selected in advance (the selection depends on C) and the number z of unoccupied selected sets is counted. The distributions of t and z under H_0 are derived and tabulated. Approximations to the power functions are obtained; they rest on the plausible but unproved assumption that t and z are approximately normally distributed.

W. Hoeffding (Chapel Hill, N. C.).

van der Vaart, H. R. Some remarks on the power function of Wilcoxon's test for the problem of two samples. I. Nederl. Akad. Wetensch., Proc. **53**, 494–506 = Indagationes Math. **12**, 146–158 (1950).

van der Vaart, H. R. Some remarks on the power function of Wilcoxon's test for the problem of two samples. II. Nederl. Akad. Wetensch., Proc. **53**, 507–520 = Indagationes Math. **12**, 159–172 (1950).

Let $X_1, \cdots, X_m$; $Y_1, \cdots, Y_n$ be samples from two populations F, G. The author comments on certain unbiasedness and symmetry properties of the test for the nonparametric two-sample hypothesis $F=G$, which is based on the signs of all differences Y_j-X_i. For very small sample sizes ($2\leqq m\leqq n\leqq 4$) he compares the power of this test with that based on Student's t against the alternatives that F and G are normal with common variance, and finds the differences to be negligible.

E. L. Lehmann (New York, N. Y.).

Finch, D. J. The effect of non-normality on the z-test, when used to compare the variances in two populations. Biometrika **37**, 186–189 (1950).

Under the assumption that two populations have the functional form of a Gram-Charlier type A series (including the kurtosis term), the author investigates the effect of nonnormality on the z-test for the comparison of two variances using the formulas developed by F. N. David [Biometrika **36**, 394–403 (1949); these Rev. **11**, 447]. A table showing the power function of the z-test is given for the sample size $n_1=25$, $n_2=61$, at the 5 per cent level of significance for various values of κ_2/κ_2', where κ_2 and κ_2' are the population variances of the two populations, both for leptokurtic and platykurtic distributions.

L. A. Aroian (Culver City, Calif.).

Baticle, E. L'interprétation des résultats d'essais sur échantillon. Génie Civil **127**, 246–248 (1950).

Discusses quality control methods using the ideas of inverse probability.

A. M. Mood (Santa Monica, Calif.).

Moran, P. A. P. A test for the serial independence of residuals. Biometrika **37**, 178–181 (1950).

In regression problems an apparent serial correlation of the dependent variable may be entirely due to serial correlation of the independent variable. To test whether this is so the author suggests testing whether there is any serial correlation among the residuals, ϵ_i, from the regression line. The testing function considered is the cyclic serial correlation coefficient $R_1=\sum_1^n \epsilon_i\epsilon_{i+1}/\sum_1^n \epsilon_i^2$; $E(R_1)$, $V(R_1)$ are derived under the assumption that the hypothesis tested is true, and it is stated that $(R_1-E(R_1))/(V(R_1))^{\frac{1}{2}}$ is asymptotically normally distributed.

D. G. Chapman.

Teatini, Ugo. Degli indici di correlazione tra caratteri ciclici. Statistica, Milano **10**, 46–67 (1950).

Reiersøl, Olav. On the identifiability of parameters in Thurstone's multiple factor analysis. Psychometrika **15**, 121–149 (1950).

Let S be the (row) vector of n test scores, X the vector of r factor scores, U the vector of n "unique parts" of the test scores, and A the $n\times r$ matrix of factor loadings. Then $S'=AX'+U'$. It is assumed that X and U are normally distributed with $EX=0$, $EU=0$, $EX'U=0$, $EX'X=Q$, $EU'U=D$ with D diagonal. The problem of identification concerns the extent to which knowledge of the distribution of the observable vector S (i.e., knowledge of $ES'S$ in this case) determines the unknown parameters (i.e., D, A, and Q in this case). The author gives a number of results most of which involve a priori conditions concerning numbers of zero coefficients in columns of A and ranks of certain submatrices of A.

T. W. Anderson (New York, N. Y.).

Banerjee, K. S. How balanced incomplete block designs may be made to furnish orthogonal estimates in weighing designs. Biometrika **37**, 50–58 (1950).

Augments the author's earlier discussion [Ann. Math. Statistics **19**, 394–399 (1948); these Rev. **10**, 201] of the connection between weighing and incomplete block designs. Numerical examples are included.

A. M. Mood.

Mathematical Biology

Geiringer, Hilda. Contribution to the heredity theory of multivalents. J. Math. Physics **26**, 246–278 (1948).

Geiringer, Hilda. Contribution to the linkage theory of autopolyploids. I. Bull. Math. Biophys. **11**, 59–82 (1949).

Geiringer, Hilda. Contribution to the linkage theory of autopolyploids. II. Bull. Math. Biophys. **11**, 197–219 (1949).

These papers deal with the mathematical genetics of polyploids. In all the assumption of chromosome segregation (rather than chromatid segregation) is made, the segregation distribution is assumed the same in both sexes, and selection, differential viability, and mutation are assumed absent. The first paper considers the case of a single locus with r alleles in a $2s$-ploid. Recurrence formulae are obtained for the distribution of gametes (and hence of genotypes), and it is shown how these may be integrated. It is shown further that unless the segregation distribution is such that maternal (and hence also paternal) genes always assort together, the limiting distribution of gametes will be

such that all genes are independently distributed. The rate of approach to the limit is examined. The second and third papers deal with the more general theory of m loci. The second paper introduces the problem of enumeration (which depends upon the detail required for the specification of the genetic constitution of the zygote). A segregation distribution is then introduced, giving the probability that at each locus an assigned number of maternal and paternal genes are transmitted. Finally, gametic recurrence formulae are found for the case $m=s=2$.

In the third paper the general gametic recurrence formula is found, and some important special cases are considered. The limit theorem of random distribution is then established, first for the case $m=s=2$, and then for the general case, subject to the condition that no exchange probability is zero. Finally, the limit behavior is investigated for the case where some exchange probabilities vanish.

C. P. Winsor (Baltimore, Md.).

Geiringer, Hilda. On some mathematical problems arising in the development of Mendelian genetics. J. Amer. Statist. Assoc. **44**, 526–547 (1949).

A summary of the probability problems arising in population genetics, largely those studied by the author [see the preceding review].

C. P. Winsor (Baltimore, Md.).

Rapoport, Anatol. Contribution to the probabilistic theory of neural nets. I. Randomization of refractory periods and of stimulus intervals. Bull. Math. Biophys. **12**, 109–121 (1950).

The author supposes first that each neuron responds once to each stimulus so long as the interval between stimuli exceeds the refractory period δ. As the frequency x of the stimulus increases, the frequency $K(x, \delta)$ of the response has the value $x, x/2, x/3, \cdots$, when $0 \leqq x < \delta^{-1}$, $\delta^{-1} \leqq x < 2\delta^{-1}$, $2\delta^{-1} \leqq x < 3\delta^{-1}$, $\cdots$. In an aggregate of neurons, varying as to δ according to the density $N(\delta)$, the "output intensity" is $f(x) = \int_0^\infty K(x, \delta) N(\delta) d\delta$. For the special case of a rectangular distribution $N(\delta)$ with limits 0 and δ_0, one obtains for large values of x the asymptotic expression $f(x) \cong \delta_0^{-1}[\log(\delta_0 x) + c]$, where c is Euler's constant. The limiting value of $f(x)$ as x becomes infinite is obtained for some other cases. For aperiodic stimuli with known statistical distribution the problem is related to that of a counter with dead time, and reference is made to Jost [Helvetica Phys. Acta **20**, 173–182 (1947); these Rev. **8**, 523]. Finally a "filter net" (band pass) is described with four neurons, including the efferent for which temporal summation is assumed. This assumption could have been avoided at the expense of replacing the single internuncial by a suitable aggregate. *A. S. Householder.*

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. V. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 23–25 (1950).

Continuing previous notes [same Bull. Cl. Sci. (5) **33**, 502–506, 718–724 (1947); **34**, 122–125, 229–231 (1948); these Rev. **9**, 297, 604; **10**, 314] the author arrives at discontinuities between species as a type of quantization arising from the occurrence of side conditions restricting the minima.

A. S. Householder (Oak Ridge, Tenn.).

Mathematical Economics

Luce, R. Duncan, and Perry, Albert D. A method of matrix analysis of group structure. Psychometrika **14**, 95–116 (1949).

The group is a social group, whose structure is defined by an irreflexive, nonsymmetric relation R (the authors use a different symbol). Two matrices X and S, and their powers, are given special attention though some others are introduced. Each element of X and S is 0 or 1; $x_{ij}=1$ in case iRj; $S_{ij}=1$ in case iRj and jRi; $x_{ii}=s_{ii}=0$ for every i. An n-chain consists of $n+1$ members i_α, $\alpha=0, \cdots, n$, such that $i_{\alpha-1}Ri_\alpha$, $\alpha=1, \cdots, n$. The chain is redundant if duplications appear. A clique is a maximal subset of 3 or more members such that $s_{ij}=1$ for all distinct i and j in the subset. If $x_{ij}^{(n)}$ and $s_{ij}^{(n)}$ denote the elements of X^n and S^n, respectively, then $x_{ij}^{(2)}$ is the number of elements j for which $s_{ij}=1$; $s_{ii}^{(2)}>0$ in case i is contained in a clique; $x_{ij}^{(n)}$ is the number of distinct n-chains of which i and j are first and last members. Since an individual i may belong to several cliques, their sizes are generally not simply related to $s_{ii}^{(2)}$. A formula is obtained expressing $s_{ii}^{(2)}$ in terms of the sizes of the cliques and of their intersections in pairs. The proofs are labored, and attempts at rigor a bit over-zealous.

A. S. Householder (Oak Ridge, Tenn.).

Luce, R. Duncan. Connectivity and generalized cliques in sociometric group structure. Psychometrika **15**, 169–190 (1950).

Continuing the investigation begun in the paper reviewed above, the author considers in particular the matrix previously denoted X, but now G. A pair (i, j) is an antimetry if $g_{ij}=1$; a symmetry if $s_{ij}=1$. If the set M of m members contains p antimetries, the set and its antimetries constitute the structure $G(m, p)$. The structure is properly n-connected in case each member is connected with each member, including itself, by a q-chain with $q \leqq n$. An n-chain is exact in case no q-chain with $q<n$ connects the same first and last members. A structure is exactly n-connected in case it contains an exact n-chain. A structure is properly n-connected if and only if all elements in $\sum_1^n G^p$ are nonnull. A subset c of M is n-connected in case every pair of distinct members is joined by a q-chain in $G(m, p)$ with $q \leqq n$, and c is an n-clique if it is also true that it is not a proper subset of a subset of M that is n-connected. A 1-clique is a clique, which modifies the definition of the previous paper. The principal theorems establish certain relations among m, p, and n in case $G(m, p)$ is n-connected. Thus if the n-connection is exact, $p \leqq (m-n-1)(m+2)+\frac{1}{2}n(n+3)$. A $G(m, p)$ that is $(2m-p)$-connected is constructed, and it is conjectured that always $n+p \geqq 2m$. At the top of p. 181 the sign "$\geqq$" should replace both equalities and "G^q" should replace "g^q." In definition 4, p. 188, "subset" should replace "subject." On p. 178, the entire 9th row of the matrix should be inverted and reversed. *A. S. Householder.*

Landahl, H. D. Note on a mechanism of the distribution of wealth. Bull. Math. Biophys. **12**, 1–6 (1950).

A mechanism of the distribution of wealth is developed under the following assumptions. There exists a large population of individuals homogeneous except for a parameter x which expresses the ability of an individual to gain or lose in exchange. The parameter is assumed to be normally distributed $p(x)$. Individuals interact only at random and in pairs. During a period of duration τ equal amounts of wealth of each is used in common, the amount contributed being a

fraction θ of the wealth of the individual having the least. If W_i represents the wealth of an individual with parameter x_i, then the total involved in an interaction is $2\theta W_i$ and the gain or loss is $\theta W_i \nu(x_j - x_i)$ where ν is a constant which determines the effectiveness of a given difference in x. The rate of production of wealth by each individual is represented by $P + AW(x) + Bx$, such that $P\ (>0)$, B, and A are constants. The consumption of wealth is given by $L + aW(x)$, where L is a minimum requirement and a is a positive constant. Hence, in an interval τ the gain in wealth of an individual (approximately $\tau \partial W(x,t)/\partial t$) may be expressed by

$$\theta\nu\int_{-\infty}^{x}(x-x')W(x',t)p(x')dx' + \theta\nu\int_{x}^{\infty}(x-x')W(x,t)p(x')dx'$$
$$+P_\tau + A_\tau W(x,t) + B_\tau x - L_\tau - a_\tau W(x,t) = \tau\partial W(x,t)/\partial t.$$

A general solution for $W(x, t)$ may now be found by standard methods. Numerical examples of the distribution of incomes for the steady state, wherein the rate of income is defined by $I(x,t) = \partial W(x,t)/\partial t + L + aW(x,t)$, are given. The results show that as $\alpha\ (=(a-A)\tau/\theta\nu)$ becomes large the distribution becomes more uniform. *M. P. Stoltz.*

Nash, John F., Jr. The bargaining problem. Econometrica **18**, 155–162 (1950).

The author considers the situations where two individuals may achieve any of a set of situations by agreement (e.g., transfer of a given item in exchange for varying possible amounts of money) but can only retain the status quo if they do not agree. Using the von Neumann-Morgenstern theory of utility, a situation may be represented by a point in a plane whose coordinates are the utilities of the two players when that situation obtains. If random combinations of situations are permitted, then the linearity properties of von Neumann-Morgenstern utility insure the convexity of the set. The utility scales may be so chosen that the utility of the status quo is zero to both players; then the origin belongs to the set S of situations possible. If $c(S)$ is the point in S finally chosen by agreement, the following assumptions are made: (1) There is no point in S better than $c(S)$ for both players; (2) if $c(T)$ is chosen from T and S is a subset of T which contains $c(T)$, then $c(S) = c(T)$; (3) if S is symmetric about a 45° line through the origin, then $c(S)$ lies on that line (i.e., bargaining skills are equal, so that symmetric conditions lead to symmetric solutions). It is then shown to follow that if S is compact, then $c(S)$ is that point in the first quadrant which maximizes the product of the utilities of the two players among all points in S. *K. J. Arrow* (Stanford University, Calif.).

Kincaid, W. M., Scott, W. R., Chover, Joshua, and Copeland, A. H. Analysis of a one-person game. Engineering Research Institute, University of Michigan, Rep. no. **M720-1 R 21.** vii+22 pp. (4 plates) (1949).

Special cases of the following one-person game are discussed. During the time interval $(-T, 0)$ certain independent Poisson distributed events (with different means) $L_1, \cdots, L_m$ may occur. The player has n cards, and on depositing a card when the event L_i occurs he gets the amount V_i, $i = 1, \cdots, m$. The problem is to devise a strategy (i.e., to specify for every t with $0 \leqq t \leqq T$ and every i with $1 \leqq i \leqq m$ whether to deposit a card if L_i occurs at $-t$) maximizing the expected gain of the player and to calculate this expected gain. More general problems are indicated. *A. Dvoretzky* (New York, N. Y.).

Dvoretzky, A., Wald, A., and Wolfowitz, J. Elimination of randomization in certain problems of statistics and of the theory of games. Proc. Nat. Acad. Sci. U. S. A. **36**, 256–260 (1950).

If $u_k(S)$, $k = 1, \cdots, p$ are completely additive nonatomic set functions defined on a Borel field $\mathfrak{S}$ of subsets of a space X and $w_j(x)$ are nonnegative $\mathfrak{S}$-measurable functions with $\sum_1^n w_j(x) = 1$, then there is a partition of X into disjoint sets $S_j \varepsilon \mathfrak{S}$ such that $\int w_j(x) du_k = u_k(S_j)$, $j = 1, \cdots, n$, $k = 1, \cdots, p$. The special case $w_1(x) = tf_1 + (1-t)f_2$, $w_2 = 1 - w_1$, where $0 < t < 1$ and f_1, f_2 are characteristic functions of $\mathfrak{S}$-measurable sets, is the theorem of Liapounoff [see Halmos, Bull. Amer. Mat. Soc. **54**, 416–421 (1948); these Rev. **9**, 574] that the range of $(u_1(S), \cdots, u_p(S))$ is convex. For games in which player I chooses an integer $i = 1, \cdots, m$, player II observes a chance variable x whose distribution depends on i, then chooses an integer $j = 1, \cdots, n$ and receives an amount a_{ij}, the above theorem implies that player II need not use mixed strategies if each distribution of x is nonatomic. Extension to games which are approximable by games of the above type and to games with several moves (sequential games) are also presented. No proofs are given. *D. Blackwell* (Stanford University, Calif.).

TOPOLOGY

***Bourbaki, N. Éléments de mathématique. X. Première partie: Les structures fondamentales de l'analyse. Livre III: Topologie générale. Chapitre X: Espaces fonctionnels; dictionnaire.** Actualités Sci. Ind., no. 1084. Hermann et Cie., Paris, 1949. ii+101 pp.

This installment of Bourbaki's treatise on abstract mathematics ends Book III, the book on general topology. The section headings are: (1) Uniform structures on functional spaces; (2) Spaces of continuous functions; (3) Equicontinuous sets; (4) Compact sets of continuous functions; and (5) Approximation of real continuous functions. If E is any set and F is a uniform space, then, with the aid of the uniform structure of F, for any given family $\mathfrak{S}$ of sets in E a uniform structure for F^E is defined. This uniform structure is called the structure of uniform convergence [of F^E] in the sets of $\mathfrak{S}$. The topology deduced from this structure is denoted by $\mathfrak{Fs}(E, F)$. Elemental properties of $\mathfrak{Fs}(E, F)$ are then derived. If F is a metric space, then known results for the set of bounded functions in $\mathfrak{Fs}(E, F)$ (when $\mathfrak{S}$ contains only the set E; in this case the structure of uniform convergence on F^E can be defined by the "écart" $d(u, v) = \sup_{x \varepsilon E}(u(x), v(x))$) are also obtained. If E is a topological space and F a uniform space, then $\mathfrak{C}(E, F)$ (the set of all continuous mappings of E into F) is next studied; $\mathfrak{C}(E, F)$ is shown to have the usual properties when considered as a subspace of $\mathfrak{Fs}(E, F)$, where $\mathfrak{S}$ consists of only E; properties for $\mathfrak{C}(E, F)$ are also obtained when regarded as a subspace of $\mathfrak{Fs}(E, F)$, with $\mathfrak{S}$ unrestricted. Attention is given to cases where E is locally compact and F is Hausdorff uniform. Equicontinuous sets are defined when E is a topological space and F is a uniform space. Various properties connected with equicontinuous sets are deduced; these include the cases of a compact E or of uniformly continuous sets. In the section on compact sets

of continuous functions, Ascoli's characterization theorem is proved for locally compact E, Hausdorff uniform F, and $\mathfrak{C}(E, F)$ when endowed with the topology of "compact convergence." Results of locally compact groups of homeomorphisms are also derived. The last section follows closely in part the exposition contained in a paper by Stone [Math. Mag. 21, 167–184, 237–254 (1948); these Rev. 10, 255]. It treats Stone's generalization of Weierstrass' approximation theorem to the effect that a subalgebra of $\mathfrak{C}(E, R)$ which contains the constant functions and separates the points of E is everywhere dense in $\mathfrak{C}(E, R)$ (E compact and $\mathfrak{C}(E, R)$ with the uniform convergence topology). One sees here the importance of the lattice-sets of functions in the theory of approximation for $\mathfrak{C}(E, R)$. Results connected with this theorem and various applications of it are given.

As was announced, a dictionary of terms (and their correspondants in English and German) used in this book is appended. *H. Tong* (Paris).

Seitz, Jiří. On a problem of E. Čech. Časopis Pěst. Mat. Fys. 75, 43–44 (1950). (Czech. English summary)

Let P be any set. A closure operator u in P defines a topology in P if (1) $u0=0$, (2) $M\subset uM$, (3) $M\subset N\subset P$ implies $uM\subset uN$. A topology u has property α if for all $M\subset P$ and all $x\varepsilon(uM)\cap M'$, there exists a set $S\subset M$ such that $uS=S\cup\{x\}$. A topology u has property β if, for all topologies v such that $vS\subset uS$ for all $S\subset P$ and $vS=S$ implies $uS=S$, the equality $uS=vS$ obtains for all $S\subset P$. It is shown that properties α and β are equivalent. *E. Hewitt* (Seattle, Wash.).

Monna, A. F. Remarques sur les métriques non-archimédiennes. I. Nederl. Akad. Wetensch., Proc. 53, 470–481 = Indagationes Math. 12, 122–133 (1950).

This paper investigates some topological properties of those metric spaces E whose distance functions ρ satisfy a non-Archimedean inequality $\rho(x, z)\leqq\max(\rho(x, y), \rho(y, z))$ for each three elements x, y, z of E (and each permutation of the labellings). It follows that each point triple of E is isosceles with vertex angle (defined by the cosine law) at most $\pi/3$. Nonnull spaces with such non-Archimedean metrics are shown to be 0-dimensional, and it is proved that in each separable 0-dimensional metric space a topologically equivalent non-Archimedean metric may be defined. It is not known whether every 0-dimensional metric space can be metrized in a non-Archimedean manner. If the elements of a metric space E constitute a field K, and one considers only topologies in E induced by valuations of K, then it is proved that a non-Archimedean metric is never equivalent to an Archimedean one (that is, one which fails to satisfy the non-Archimedean inequality for at least one triple of elements). In the last section of the paper the notion of non-Archimedean metric is introduced in uniform spaces which are not necessarily metric. *L. M. Blumenthal.*

Monna, A. F. Remarques sur les métriques non-archimédiennes. II. Nederl. Akad. Wetensch., Proc. 53, 625–637 = Indagationes Math. 12, 179–191 (1950).

A uniform structure is defined in a topological space E by a family $\mathfrak{F}$ of subsets U of the product space $E\times E$ with the properties (1) πU is the diagonal D (the set (x, x), $x\varepsilon E$), 2) the product of two sets of $\mathfrak{F}$ contains a set of $\mathfrak{F}$, (3) $U\varepsilon\mathfrak{F}$ implies $U^{-1}\varepsilon\mathfrak{F}$, (4) $U\varepsilon\mathfrak{F}$ implies the existence of $V\varepsilon\mathfrak{F}$ such that $\overset{2}{V}\subset U$; where U^{-1} consists of those elements (y, x), $x, y\varepsilon E$, such that $(x, y)\varepsilon U$, and $\overset{2}{V}$ denotes the set of elements (x, y) of $E\times E$ having the property that an element z of E exists such that (x, z) and (z, y) belong to V. Defining a neighborhood $U(x)$ ($x\varepsilon E$) to consist of those points y of E such that $(x, y)\varepsilon U$ provides a topology for E, and E is called uniform. A uniform structure is non-Archimedean if and only if for each U of $\mathfrak{F}$, $\overset{2}{U}\subset U$. The author shows that uniform spaces with non-Archimedean structures are 0-dimensional and homeomorphic to a subset of a product of metric spaces, with the metric in each space non-Archimedean. A necessary and sufficient condition that a non-Archimedean uniform space be isomorphic with a non-Archimedean metric space is obtained. The topology of a Hausdorff space is called non-Archimedean if a complete system S of neighborhoods exist such that if $U_x, U_y\varepsilon S$ then either $U_x\cdot U_y=0$ or one of the neighborhoods is a subset of the other. It is shown that each Hausdorff space with non-Archimedean topology is 0-dimensional. In conclusion, the author proves that each separable metric space of dimension exceeding 1 contains infinitely many non-Archimedean triples and he poses the problem of characterizing those 1-dimensional metric spaces that may be metrized (with preservation of the topology) so as to contain no non-Archimedean triples. [Reviewer's remark. It may be shown that any compact metric space which contains no non-Archimedean triples is homeomorphic to a bounded and closed subset of the Euclidean line. If the compact metric space is also 1-dimensional then it is, of course, homeomorphic with a segment.] *L. M. Blumenthal* (Columbia, Mo.).

Medgyessy, P. Über nichtprimitive reguläre Graphen dritten Grades. Publ. Math. Debrecen 1, 183–185 (1950).

The author considers any connected regular graph of degree 3 in which the edges are arranged in two classes A and B so that each vertex of the graph is incident with just one edge of class B. He demonstrates the existence of a re-entrant path in G in which edges of classes A and B occur alternately, and which describes each edge of class A just once and each edge of class B just twice. *W. T. Tutte.*

Bassi, Achille. Sul concetto di complesso e di equivalenza combinatoria. Ann. Mat. Pura Appl. (4) 28, 327–344 (1949).

The usual definition of topological complex is modified in such a way that the closure of a cell need not be the homeomorphic incidence-preserving image of a convex polyhedral complex. Two complexes are called equivalent if one is reducible to the other by a finite number of sectioning operations, appropriately defined, and their inverses. The author exhibits two complexes Σ and K (Σ being polyhedral) which are homeomorphic but which are not equivalent in the classical sense. These complexes are equivalent in the new sense but Σ cannot be reduced to a complex combinatorially equivalent to K by elementary hyperplanar subdivisions and their inverses. *P. A. Smith.*

Fary, István. Remarque sur le prolongement des transformations topologiques. Publ. Math. Debrecen 1, 109–115 (1949).

The following theorem has been proved by Alexander [Proc. Nat. Acad. Sci. U. S. A. 10, 6–8 (1924)]. Let S be a polyhedron in E^3 homeomorphic to a 2-sphere. Then the union of S and its interior is homeomorphic to the solid ball B given by $x^2+y^2+z^2\leqq 1$. Alexander showed that the same conclusion holds if S is a surface in E^3 homeomorphic to S^2 with a certain condition on the intersection of S with each

plane of a parallel family. The author proves a theorem of this kind with a hypothesis in terms of the paratingent. At a point p of S the paratingent is the set of all lines which are limits of secant lines through points x and y in S which approach p. The main result is then the following. Let S be a surface in E^3 which is homeomorphic to S^2. If at each point of S the paratingent omits at least one line then the union of S and its interior is homeomorphic to B.

D. Montgomery (Princeton, N. J.).

Borsuk, Karol. On the third symmetric potency of the circumference. Fund. Math. **36**, 236–244 (1949).

The nth symmetric product of a space X, which we denote by $X^{(n)}$, is the collection of all unordered subsets of X containing n or fewer points. It is thus a subset of the space 2^X of all closed subsets of X, and the topology of $X^{(n)}$ is that which it inherits from 2^X. In this paper it is shown that the third symmetric product $S^{(3)}$ of the circumference S is topologically equivalent to the Cartesian product of the circumference and the 2-sphere. The proof is carried out by dissecting $S^{(3)}$ into a number of simple pieces and then putting them back together one at a time.

E. G. Begle (New Haven, Conn.).

***Kiang, Tsai-han. On certain groups of automorphisms of Stiefel manifolds $V_{n,m}$ and properties of the ground manifolds.** National Peking University Semi-Centennial Volume, Mathematical, Physical and Biological Series, pp. 45–50, 1948.

The Stiefel manifold $V_{n,m}$ is the set of all ordered and orthogonal systems $v_{n,m}=(v_1, v_2, \cdots, v_m)$, $0<m<n$, $2<n$, of m unit vectors at a point of Euclidean n-space. If we identify all the 2^m systems $(\epsilon_1 v_1, \epsilon_2 v_2, \cdots, \epsilon_m v_m)$, where $\epsilon_i=\pm 1$, we obtain the manifold $W_{n,m}$, which is the object of study of the present note. Such manifolds are of interest in the study of nonoriented linear elements of manifolds. The main result is that $W_{n,m}$, if $n-m>1$, is simple in dimension $n-m$ if $n-m$ is odd, while it is not $(n-m)$-simple if $n-m$ is even. It is also shown that $W_{n,m}$ is orientable or not according as $n-m$ is odd or even. *E. G. Begle.*

Cartan, Henri. Une théorie axiomatique des carrés de Steenrod. C. R. Acad. Sci. Paris **230**, 425–427 (1950).

The theory of $\cup_i$-products of Steenrod [Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154] is carried over from simplicial to more general complexes. Having such products in (chain) complexes A and B, a similar product is defined in the tensor product $A\otimes B$ by an explicit formula. This is used to introduce the Steenrod products in the author's cohomology theory of topological spaces based on gratings. As an application the author states the formula $\mathrm{Sq}_i(\xi\cup\eta)=\sum_{j+k=i}\mathrm{Sq}_j(\xi)\cup\mathrm{Sq}_k(\eta)$ connecting the Steenrod squares with the ordinary $\cup$-product for (mod 2) cohomology classes ξ, η. No proofs are given. *S. Eilenberg.*

Thom, René. Classes caractéristiques et i-carrés. C. R. Acad. Sci. Paris **230**, 427–429 (1950).

Let E be a space fibered by spheres S^{k-1}, whose base space is a complex K with cells Z_i. Filling in the spheres gives a space A, which may be cut into cells $Z_i\times b^k$. To the transformation $Z_i\times b^k\to Z_i$ corresponds an isomorphism φ^* of the cohomology group $H^r(K)$ onto $H^{r+k}(A')$ ($A'=A$ mod E). If I^0 is the unit cocycle of K, then $U^k=\varphi^* I^0$ corresponds to W^k in K; the W^r are the characteristic classes of the bundle. Also, (1) $\varphi^* W^k=U^k\cup U^k$. If the fibering is simple, then (2) $\mathrm{Sq}^j\varphi^*=\varphi^*\mathrm{Sq}^j$ [Sq^j=Steenrod square; see Ann. of Math. (2) **50**, 954–988 (1949); these Rev. **11**, 122]. The main formula is:

$$\mathrm{Sq}^r U^k=\varphi^* W^r, \qquad r=0, 1, \cdots, k. \tag{3}$$

For $r=k$, this is (1). For $r<k$, there is a set of $k-r$ independent vectors in the fibers over K^{k+r}, which can be used to express the bundle over K^{k+r} as a product of a simple bundle with one whose fibers are of dimension $k-r$ less than before; (1) and (2) are now used. *H. Whitney.*

Thom, René. Variétés plongées et i-carrés. C. R. Acad. Sci. Paris **230**, 507–508 (1950).

For a smooth manifold M^p in a smooth manifold M^n, there is a homomorphism ψ^* of $H^n(M)$ into $H^{n-p+r}(V)$ (the Gysin operation) corresponding to φ^* [see the preceding review]. If u^{n-p} is the cocycle corresponding to M in V, then $\mathrm{Sq}^r u^{n-p}=\psi^* W^r$ (W^r for the normal bundle). If no nonbounding cycle of M bounds in V, then ψ^* is one-to-one, and the W^r are determined by the homology properties of M in V. This is true, in particular, for the diagonal M of V in $V\times V$. Moreover, in this case, there is a correspondence between the normal and tangent bundles. Hence the W^k for a tangent bundle depend only on the homology structure of the manifold, not on the differential properties. If Δ is the cocycle of the diagonal, $\mathrm{Sq}^i\Delta=\psi^* W^i$. *H. Whitney.*

Wu, Wen-tsün. Classes caractéristiques et i-carrés d'une variété. C. R. Acad. Sci. Paris **230**, 508–511 (1950).

Since the Steenrod squares are homomorphisms, one may define in an n-manifold "canonical classes" U^p ($0\leqq 2p\leqq n$) such that $U^p\cup Y^{n-p}\backsim\mathrm{Sq}^p Y^{n-p}$ (all Y^{n-p}). Then in a smooth manifold the $W^i=\sum_p \mathrm{Sq}^{i-p} U^p$ are the characteristic classes of the tangent bundle. The proof depends on results of Thom [see the two preceding reviews] and formulas of H. Cartan [see the third preceding review]. As application, some results on particular W^i are given. *H. Whitney.*

Wu, Wen-tsün. Les i-carrés dans une variété grassmannienne. C. R. Acad. Sci. Paris **230**, 918–920 (1950).

The characteristic classes W^i of a sphere bundle satisfy

$$\mathrm{Sq}^r W^s=\sum_t\binom{s-r}{t} W^{r-t}W^{s+t}, \qquad 0<r\leqq s,$$

coefficients always mod 2. Some consequences about the W^i for the tangent bundle of a manifold are given.

H. Whitney (Cambridge, Mass.).

Chern, Shiing-shen, and Spanier, E. The homology structure of sphere bundles. Proc. Nat. Acad. Sci. U. S. A. **36**, 248–255 (1950).

Let the space X be fibered into spheres S of dimension d; let the base space B be a complex. Let f be the projection of X onto B; it defines a homomorphism $f^*: H^p(B)\to H^p(X)$. There is a homomorphism $\Phi: H^p(X)\to H^{p-d}(B)$, corresponding to the inverse of Steenrod's functional cup product. Let Ω be the basic characteristic $(d+1)$-class of the bundle; using cup products with Ω defines $\Psi: H^{p-d}(B)\to H^{p+1}(B)$. The main theorem is that the cohomology sequence with these mappings is exact. It is noted that this theorem was discovered independently by R. Thom [see the fourth preceding review]. Some applications are given, including results of J. Leray, G. Hirsch, and A. Lichnerowicz, as follows. If S bounds in X, then $k\Omega\neq 0$ for each $k\neq 0$; then d must be odd. If S does not bound, $H^p(B\times S)\approx H^p(X)$. Some relations between the Poincaré polynomials of B, X, and S are given.

H. Whitney (Cambridge, Mass.).

Komatu, Atuo. Relations between homotopy and homology. II. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 1, 46–55 (1950).

This note is a continuation of a previous paper by the same author [Osaka Math. J. 1, 150–155 (1949); these Rev. 11, 122]. The author's main object is to give new explanations of some ideas introduced by H. Hopf [Comment. Math. Helv. 17, 307–326 (1945); these Rev. 7, 36], such as "Homotopie-Ränder" and "Henkel cycles."

W. S. Massey (Providence, R. I.).

Burger, E. Über Schnittzahlen von Homotopieketten. Math. Z. 52, 217–255 (1949).

Pontrjagin's definition [Math. Ann. 105, 165–205 (1931)] of pairs of Abelian groups is generalized to pairs of Abelian groups $\mathfrak{G}$, $\mathfrak{H}$ with operators, the operator domain being the ring $\mathfrak{X}$ of integers in an algebraic number field K. The product $g \cdot h$ is required to be distributive and to satisfy a condition generalizing the Pontrjagin condition of primitivity; in addition, the presence of the operators calls forth a new condition, $(x_1 g) \cdot (x_2 g) = x_1 (g \cdot h) \bar{x}_2$, where $g \in \mathfrak{G}$, $h \in \mathfrak{H}$, $x_1, x_2 \in \mathfrak{X}$ and $\bar{x}$ denotes the conjugate of x. A generalized multiplication is called an intersection relation if the value domain is $\mathfrak{X}$, and a linking relation if the value domain is $K \bmod 1$. The generalization of the primitivity concept differs in these two cases. The analysis of these two multiplications is based on the Steinitz theory of matrices with entries from K [Math. Ann. 71, 328–354 (1911); 72, 297–345 (1912)]. In order that an intersection relation exist it is necessary that the operator groups $\mathfrak{G}$ and $\mathfrak{H}$ be of modul type, that they have the same rank, and that their classes be conjugate reciprocals. In order that a linking relation exist it is necessary that $\mathfrak{G}$ and $\mathfrak{H}$ be finite and that the invariants of $\mathfrak{H}$ be the conjugates of the invariants of $\mathfrak{G}$. Thus, in either case, the structure of $\mathfrak{H}$ is determined by that of $\mathfrak{G}$. If either kind of relation exists it is unique up to automorphisms of $\mathfrak{H}$. These theorems are generalizations of Pontrjagin's duality theorem.

The familiar intersection numbers and linking numbers are instances of these relations in the special case that $\mathfrak{X}$ is the ring of rational integers in the field K of rational numbers. Then $\mathfrak{G}$ and $\mathfrak{H}$ are the integral Betti or torsion groups of a complex of the appropriate dimensions. More significant is the application of this theory to Reidemeister's homotopy chains [Monatsh. Math. Phys. 48, 226–239 (1939); these Rev. 1, 105]. Then $\mathfrak{G}$ and $\mathfrak{H}$ are Betti or torsion groups with $\mathfrak{X}$ as coefficient domain. The Reidemeister τ-multiplication is shown to be an intersection relation and the ρ-multiplication a linking relation. (This application is, of course, the motivation of the whole paper.)

If $\mathfrak{G}$ and $\mathfrak{H}$ are identical, e.g., in an n-dimensional manifold the Betti group of dimension $\frac{1}{2}n$ or the torsion group of dimension $\frac{1}{2}(n-1)$ (with $\mathfrak{X}$ as coefficient domain), one has generalizations of self-intersection numbers and self-linking numbers. The motivating application suggests that the usual requirements of symmetry or skew-symmetry be generalized to "Hermitian" or "skew-Hermitian" character. The self-intersection and self-linking relations in $\mathfrak{G} = \mathfrak{H}$ are generally not unique; on the contrary, as applied to homotopy chains they lead to new topological invariants. The author does not pursue this line of investigation (which may be quite difficult) but is content with an example showing how certain 5-dimensional lens spaces may be distinguished by means of Reidemeister's ρ-multiplication.

R. H. Fox (Princeton, N. J.).

Whitehead, J. H. C. A certain exact sequence. Ann. of Math. (2) 52, 51–110 (1950).

A detailed exposition of results announced in a note [Proc. Nat. Acad. Sci. U. S. A. 36, 55–60 (1950); these Rev. 11, 379]. In addition to this he constructs an exact sequence $S(X)$ [see the cited review] for an arbitrary C^0 space by construction of a polytope with the same homotopy structure as X. In an appendix, he also shows that a C^0 space "dominated" by a countable CW polytope belongs to the homotopy type of a locally finite polytope.

J. Dugundji (Los Angeles, Calif.).

GEOMETRY

Favard, Jean. Sur l'axiome de Pasch considéré comme axiome d'espace. C. R. Acad. Sci. Paris 230, 1996–1997 (1950).

Adjoin to a postulation system which assumes merely (1) each two "points" are on one and only one "line," and (2) Hilbert's postulates of order for points of a line, the Pasch axiom in the form: if a, b, c are any three points not on a line, c' a point of the line $L(a, b)$ not between a and b, and b' a point of $L(a, c)$ between a and c, then $L(b', c')$ intersects $L(b, c)$ in a point a' between b and c. The author asserts that in this system "plane" can be defined in the following manner. If a, b, c are points not on a line, let A denote the point set $L(a, b)+L(a, c)+L(a, a')$, where $a' \in L(b, c)$ and a' is between b and c. Defining point sets B and C analogously, then plane $P(a, b, c) = A+B+C$. It is clear how this procedure can be extended to define higher dimensional subspaces. No proofs are offered.

L. M. Blumenthal (Columbia, Mo.).

Sperner, Emanuel. Beziehungen zwischen geometrischer und algebraischer Anordnung. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949, no. 10, 413–448 (1949).

Detailed development of results announced in an earlier publication [Arch. Math. 1, 148–153 (1948); these Rev. 10, 729].

F. A. Behrend (Melbourne).

Haupt, Otto. Über die Verteilung der Relativordnungen bezüglich eines Bogens. Math. Z. 52, 527–546 (1950).

Let arc $\mathfrak{C}$ denote the continuous image in projective n-space of the closed interval. Secant denotes the linear $(n-1)$-space which cuts $\mathfrak{C}$ at every common point; the intersection of $\mathfrak{C}$ with a t-secant has exactly t original points in the parameter interval $[t \geqq 1]$. Order k of $\mathfrak{C}$ denotes the maximum number of parameter values which correspond to sets of collinear points of $\mathfrak{C}$. In the following, the existence of a finite k is assumed. (Relative) order of the point X with respect to $\mathfrak{C}$ is defined as the largest t for which there are t-secants through X. Call P an ordinary point if every X of some neighborhood of P has an order which is not smaller than that of P. A normal point is an interior point of the set of all the ordinary points; $\mathfrak{Q}(r)$ is the complement of the set of all the normal points of order not less than r $(1 \leqq r \leqq k)$.

The existence of points of the largest possible order k is proved. The points of the orders $k-1$ and k are necessarily normal. The sets $\mathfrak{Q}(r)$ are closed. The set $\mathfrak{Q}(1)$ of all the nonnormal points lies on $\mathfrak{C}$ and is nowhere dense in $\mathfrak{C}$, and $\mathfrak{Q}(k)$ is projectively equivalent to a bounded set. Each component of each $\mathfrak{Q}(r)$ is projectively equivalent to a convex body. If $\mathfrak{C}$ is the sum of a finite number of convex

arcs in the projective plane, then the boundary of each component is a convex polygon. If the two end-points of $\mathfrak{C}$ do not coincide, then one of the two connecting straight line segments lies in some component of $\Omega(k)$. Given an arc of order n connecting these two end-points and which lies in the same component, then the union of $\mathfrak{C}$ with this arc is a closed curve of order $\leqq k+2[\frac{1}{2}(n-1)]$. The author's main tool is the following "reduction theorem." Suppose there are r parameter values whose images on $\mathfrak{C}$ lie in an $(n-1)$-space. Then the latter is the limit of t-secants where $t \geqq r$ [cf. Monatsh. Math. Phys. **40**, 1–53 (1933)]. The various concepts and results are illustrated by many examples. *P. Scherk* (Saskatoon, Sask.).

✶**Kommerell, Karl. Vorlesungen über analytische Geometrie des Raumes.** 2d ed. K. F. Koehler Verlag, Stuttgart, 1950. viii+407 pp. DM 26.00.

New edition, with relatively few changes, of a book first published in 1940 [cf. these Rev. **2**, 9]. Table of contents: (1) Das Koordinatensystem, Ebene und Gerade, Kugel; (2) Flächen und Raumkurven; (3) allgemeine Eigenschaften der Flächen zweiter Ordnung, quadratische Formen, Invarianten; (4) Klassifikation der Flächen zweiter Ordnung, Kreisschnittebenen; (5) besondere Eigenschaften der einzelnen Flächen zweiter Ordnung; (6) Tetraederkoordinaten; (7) Kollineationen und Korrelationen; (8) das Flächenbüschel zweiter Ordnung.

✶**Heffter, Lothar. Grundlagen und analytischer Aufbau der projektiven, euklidischen, nichteuklidischen Geometrie.** 2d ed. B. G. Teubner, Leipzig, 1950. 192 pp. $3.00.

This is essentially the same as the first edition [1940; these Rev. **2**, 9] with some small errors corrected and a few new paragraphs added at the end. The book is divided into five parts. Part A, "Foundations," deals chiefly with real projective 3-space. It begins with a set of twenty axioms of incidence, arranged in space-dual pairs [cf. K. Menger, Duke Math. J. **17**, 1–14 (1950); these Rev. **11**, 533], three axioms of order, and Dedekind's axiom of continuity. These are followed by definitions of abscissa and cross ratio, including the useful concept of mixed cross ratio (of two points and two lines or planes). In Part B, "Projective geometry," the scope of the projective line is extended by allowing the abscissa of a point to be complex. After defining ordinary homogeneous coordinates x_1, x_2, whose ratio is the abscissa, the author defines contragredient coordinates u_1, u_2, such that $u_1x_1+u_2x_2=0$, and deduces an elegant expression for cross ratio. These ideas are extended to two and three dimensions and pursued as far as the projective classification of conics and quadrics. In Part C, "Parallel geometry," affine geometry is derived by singling out a plane to serve as plane at infinity, and the classification of conics and quadrics is continued. In Part D, "Orthogonal geometry," Euclidean geometry is derived by singling out an imaginary conic in the plane at infinity. In Part E, "Non-Euclidean geometry," elliptic and hyperbolic geometry are derived from projective geometry by singling out a nondegenerate quadric, imaginary or oval, respectively. Because of its emphasis on analytic methods, this part may be roughly described as a condensed version of Klein's "Vorlesungen über nicht-euklidische Geometrie" [Springer, Berlin, 1928]. The new material [on pp. 183–187] consists of an expanded and illustrated account of the Euclidean models wherein distances along lines through one fixed point (as well as angles between pairs of these lines) are represented without distortion. *H. S. M. Coxeter.*

Pöttker, Werner. Die Übertragung der Guldinschen Regeln auf den sphärischen und elliptischen Raum. Arch. Math. **2**, 192–198 (1950).

If a plane arc (or area) revolves about an axis in its plane, not intersecting it, the surface (or volume) generated is equal to the length (or area) multiplied by the length of the path of its centroid. As usual, the author attributes these theorems to Guldin, though they are really due to Pappus [see W. W. R. Ball, A Short Account of the History of Mathematics, 5th ed., Macmillan, London, 1912, p. 101]. Using quaternions, he proves that they continue to hold in spherical or elliptic space when the centroid is suitably defined. The arc (or area) is regarded as lying on a sphere (i.e., hypersphere) in Euclidean 4-space, and its ordinary centroid is projected centrally onto the sphere.

H. S. M. Coxeter (Toronto, Ont.).

Fulton, Curtis M. The non-Euclidean mirror. Proc. Amer. Math. Soc. **1**, 331–333 (1950).

Courant and Robbins [What is Mathematics?, Oxford University Press, New York, 1941; these Rev. **3**, 144] gave a proof that any curve reflecting rays through one fixed point into rays through another fixed point is an ellipse. The author remarks that this holds just as well in hyperbolic geometry as in Euclidean. (In hyperbolic geometry an ellipse may be defined either by its constant sum of focal distances or as a conic interior to the absolute.) He gives also an analytic proof, followed by a similar treatment of the parabola reflecting rays through a fixed point into rays parallel to a fixed line. This analysis could have been avoided by regarding the parabola as the limiting form of an ellipse when one focus recedes to infinity, or as a conic having simple contact with the absolute. The interesting fact is that such a parabola still has a focus, though its "directrix" is not a straight line but a horocycle.

H. S. M. Coxeter (Toronto, Ont.).

✶**Steinhaus, H. Mathematical Snapshots.** Oxford University Press, New York, N. Y., 1950. vi+266 pp. $4.50.

The first English edition was published in 1938 [Stechert, New York, N. Y.]. In this edition additions to the text have been made and the number of illustrations increased considerably. The anaglyphs of the first edition have been replaced by photographs.

Touchard, Jacques. Sur un problème de configurations. C. R. Acad. Sci. Paris **230**, 1997–1998 (1950).

Sainte-Laguë [Les réseaux . . . , Gauthier-Villars, Paris, 1926, p. 39] a défini, pour toute permutation de degré m, un schéma composé d'une droite coupée par une courbe en m points. L'auteur étudie les demi-courbes séparées par l'axe du schéma. Ce sont des configurations homéomorphes à celle-ci. Soient, sur un axe, $2n$ points consécutifs équidistants. Dans l'un des demiplans, on joint ces points deux à deux par n demi-cercles. Parmi les $1\cdot3\cdot5\cdot7\cdots(2n-1)$ configurations obtenues combien y en a-t-il qui aient p points doubles? Le cas général est ramené à celui où tous les arcs sont connexes (aucune perpendiculaire à l'axe ne séparant alors le système en deux parties disjointes). Si x^p représente une configuration avec p points doubles, les configurations formant des systèmes uniques seront repré-

sentées par un polynome $S_n(x)=R(a_1, a_2, a_3, \cdots, a_n)$; $a_p=1+x+x^2+\cdots+x^{p-1}$. Le polynome R est homogène, de degré n, et peut être construit de proche en proche. Sa fonction génératrice est une fraction continue que l'auteur rattache aux fonctions θ. D'autres propriétés des systèmes uniques sont annoncées. Page 1998, ligne 1, lire $1\cdot3\cdot5\cdots$ au lieu de $1, 3, 5, \cdots$. Si $p=0$, on retrouve un résultat obtenu par A. Errera [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **11**, no. 6 (1931)]. *A. Sade* (Marseille).

ApSimon, Hugh. Three facially-regular polyhedra. Canadian J. Math. **2**, 326–330 (2 plates) (1950).

A polyhedron is said to be uniform if its faces are regular polygons and it admits symmetries which transform a given vertex into every other vertex in turn. The author calls a polyhedron facially-regular if it is uniform and has all its faces congruent. He describes three facially-regular polyhedra, each infinite in three dimensions. Their faces are equilateral triangles. In the three cases, the number of triangles meeting at each vertex is 12, 9, and 8.

W. T. Tutte (Toronto, Ont.).

Carver, W. B. The problem of eight points. Amer. Math. Monthly **57**, 307–316 (1950).

Le problème des huit points est posé dans le "Traité de Topographie d'Arpentage et de Nivellement" de Puissant [Paris, 1820, p. 55]. Quatre points A, B, C, D sont vus de quatre autres points E, F, G, H. Les seules données étant les angles sous lesquels chacun des segments AB, BC, CD est vu de chacun des points E, F, G, H, on demande de trouver les distances relatives de ces huit points. L'auteur expose la solution de Puissant, en fait une discussion sommaire, signale deux transformations de Cremona liées au problème et suggérées par la recherche d'exemples numériques simples et donne enfin des exemples correspondant à divers cas particuliers. *M. Decuyper* (Lille).

Popov, B. S. Contribution à la géométrie du triangle. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2**, 111–134 (1949). (Macedonian. Russian and French summaries)

Well-known properties in the geometry of the triangle, like the properties of the centroid, Euler's theorem on the distance between the circumcenter and the incenter, etc. are proved by the use of vectors, in order to exhibit the advantages of the method in this connection. Such an application of vectors is neither a novelty nor a rarity.

N. A. Court (Norman, Okla.).

Ciani, Edgardo. Coniche notevoli di un fascio. Atti. Accad. Ligure **20**, 15–18 (1942).

Ničе, Vilko. Démonstration et complément d'un théorème concernant le conoïde de Plücker. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 173–175 (1949). (Croatian. French summary)

Marussi, Antonio. Sulle formule di Helmert e sui metodi per la trasformazione di reti sull'ellissoide. Ann. Triestini. Sez. 2. (4) **1(17)**, 85–130 (1947).

Lehmer, D. H. Approximations to the area of an n-dimensional ellipsoid. Canadian J. Math. **2**, 267–282 (1950).

In several dimensions the ellipsoid is approximated as to area by a sphere whose radius is some type of average of the semi-axes. The best average is determined in the general case where the eccentricities are small, as well as for those special well poised ellipsoids for which a certain coefficient vanishes. *P. Franklin* (Cambridge, Mass.).

Convex Domains, Extremal Problems, Integral Geometry

Stone, M. H. Postulates for the barycentric calculus. Ann. Mat. Pura Appl. (4) **29**, 25–30 (1949).

Let K denote a set of elements $x, y, \cdots$ called points and Γ an ordered division ring whose elements $\alpha, \beta, \cdots$ can be associated with the points of K as masses. The author develops an abstract theory of barycenters and, hence, an abstract theory of convex sets. The theory is based on a set of postulates in terms of a function $(x, y; \alpha, \beta)$, with values in K, of two points x, y and associated nonnegative masses α, β. The point $(x, y; \alpha, \beta)$ is called the barycenter of the masses α, β placed at the points x, y. The postulates are the following. (I) $(x, y; \alpha, \beta)$ is defined for all x, y in K and all nonnegative α, β for which $\alpha+\beta>0$. (II) $(x, y; \alpha, 0)=x$. (III) $(x, x; \alpha, \beta)=x$. (IV) $(x, y; \lambda\alpha, \lambda\beta)=(x, y; \alpha, \beta)$ for $\lambda>0$. (V) $(x, z; \alpha, \beta)=(y, z; \alpha, \beta)$ and $\alpha>0$ imply $x=y$. (VI) $((x, y; \alpha, \beta), z; \alpha+\beta, \gamma)=((z, y; \gamma, \beta), x; \gamma+\beta, \alpha)$. The barycenter of an arbitrary finite system of masses is defined by induction and proved to be independent of the order in which the masses are taken. The main result of the paper is the following imbedding theorem. There exists a vector space M over Γ such that K is a convex subset of M and $z=(x, y; \alpha, \beta)$ is equivalent to $(\alpha+\beta)z=\alpha x+\beta y$.

W. Fenchel (Princeton, N. J.).

Knothe, Herbert. Über eine Vermutung H. Minkowskis. Math. Nachr. **2**, 380–385 (1949).

Minkowski's conjecture that the equality sign in the inequality $S^3 \geqq MV$ (V volume, S surface area, M integral of the mean curvature of a convex body K in the ordinary space) is only valid if K is a cap body of a sphere has been proved by G. Bol [Abh. Math. Sem. Hansischen Univ. **15**, 37–56 (1943); these Rev. **7**, 474]. The author outlines another proof which avoids inner parallel bodies. [However, the reviewer is unable to follow the author's arguments because of (possibly unessential) errors and gaps. The formulation of the definition of the convergence of a sequence of convex bodies is wrong. The assumption that a quantity is $o(\epsilon^3)$ does not imply that it is greater than or equal to $c\epsilon^2$, c a positive constant. The main lemma is concerned with a family of convex bodies with the same volume. In order to apply it to a family satisfying the other assumptions of the lemma, but not this, the bodies are normalized by similarities. It is not shown that the other assumptions are preserved under this normalization.] *W. Fenchel.*

Schüler, Hans. Vereinfachter Beweis eines Minkowskischen Satzes über konvexe Körper mit Mittelpunkt. Arch. Math. **2**, 202–204 (1950).

The author makes an error in attempting to give a short proof of Minkowski's theorem on the successive minima of a bounded convex body symmetric in the origin. The difficulty centers around the following erroneous statement on page 204: "$\mathfrak{B}^{(i)}$ ist . . . ganz enthalten in dem Körper $\mathfrak{B}^{(i)*}$." *P. T. Bateman* (Urbana, Ill.).

Schneider, Theodor. Verallgemeinerung einer Minkowskischen Ungleichung über konvexe Körper mit Mittelpunkt. Math. Ann. **122**, 35–36 (1950).

This paper contains the same fallacy as the paper reviewed above. *P. T. Bateman* (Urbana, Ill.).

Bertolini, Fernando. Su alcune superficie ad area minima. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 254–267 (1949).

Minimal surfaces of revolution furnish a classical example in the calculus of variations; the discussion is usually given in terms of Cartesian coordinates, with meridian curve referred to a parameter whose range is the axis of revolution. The discussion is now given quite simply in terms of cylindrical coordinates (r, θ, z), with meridian curve $z=f(r)$ referred to a parameter whose range is normal to the axis of revolution. Also discussed are surfaces of revolution $z=f(r)$ of minimum area subject to the restriction of being tangent at the boundary to a fixed right circular cone and satisfying $|f''(r)| \leqq L(r)$, where $L(r)$ is a given nonnegative continuous function; the latter discussion is shown to extend more generally to surfaces $z=F(r, \theta)$. *E. F. Beckenbach.*

Bieri, H. Ein isoperimetrisches Problem mit Nebenbedingung. Experientia **6**, 222–224 (1950).

The author considers the class of all convex bodies of revolution which have a given volume V and whose axes have a given length l. He determines the body in the class which has maximum surface area. This body is uniquely determined and is a cylinder, a truncated cone, or a cone according as $V \leqq \pi l^3/36$, $\pi l^3/36 < V < \pi l^3/24$, or $\pi l^3/24 \leqq V$. *W. Fenchel* (Princeton, N. J.).

Süss, Wilhelm. Über Eibereiche mit Mittelpunkt. Math.-Phys. Semesterber. **1**, 273–287 (1950).

The paper discusses characteristic properties of plane convex domains with a center. Let E be a plane, closed, bounded, and convex set with interior points. Let P be an arbitrary point of E and $s(P)$ a chord through P. Denote by $f(s(P))$ the smallest of the areas of the two sets E' and E'' into which $s(P)$ divides E. For fixed P and varying s this function has a minimum $F(P)$. A chord for which this minimum is attained is called a minimal chord through P. It is shown that if there is at most a countable number of points of E with more than one minimal chord, then E has a center. The proof is based on an investigation of the level sets of the function $F(P)$. Consider the areas of the maximal triangles inscribed in E' and E'' and having $s(P)$ as a side. Let $f(s(P))$ now denote the smallest of these two numbers. If the boundary of E does not contain segments and if there is at most one point with more than one minimal chord in this new sense, then E has a center and belongs to the class of convex domains with conjugate diameters studied by J. Radon [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math. Phys. Kl. **68**, 123–128 (1916)]. Finally the function $f(s(P))=\min(s'/s, s''/s)$ is considered where s, s', s'' are the lengths of the chord $s(P)$ and of the segments into which it is divided by P. Let $F(P)$ denote the minimum of f for fixed P. Assume that the boundary of E does not contain segments. Then E has a center if and only if there is one and only one point P in E such that there are at least two chords through it for which $f(s(P))=F(P)$.

W. Fenchel (Princeton, N. J.).

Egerváry, E. On the smallest convex cover of a simple arc of space-curve. Publ. Math. Debrecen **1**, 65–70 (1949).

In the ordinary space consider a simple arc γ with continuously varying tangent. Let $\Omega(\gamma)$ be the convex hull of γ. It is shown that if (and, obviously, only if) γ is of order 3, i.e., no 4 points of γ are coplanar, $\Omega(\gamma)$ is filled up simply by the chords of γ such that every interior point of $\Omega(\gamma)$ is on one and only one chord of γ. For an arc $\gamma=\overset{\frown}{AB}$ of this kind, the volume of $\Omega(\gamma)$ turns out to be $\frac{1}{3}$ the chord AB times the area of the projection of γ on a plane normal to AB. By means of this result the following isoperimetric problem is solved. Among all arcs γ of order 3 with a given length to find the arc for which the volume of $\Omega(\gamma)$ is maximum. The solution is an arc of a helix uniquely determined by the given length. *W. Fenchel* (Princeton, N. J.).

Inzinger, Rudolf. Stützbare Bereiche, trigonometrische Polynome und Defizite höherer Ordnung. Monatsh. Math. **53**, 302–323 (1949).

A supportable domain (better: curve) H is a closed plane curve such that for every φ, $0 \leqq \varphi < 2\pi$, there is one and only one oriented tangent with the direction φ. The curve has to be considered as the envelope of these tangents. The supporting function $H(\varphi)$ of H is the distance of this tangent from the origin and is assumed to have period 2π and to be continuous and piecewise continuously differentiable. Conversely, every periodic function $H(\varphi)$ with these properties determines a supportable domain. [Cf. H. Geppert, Math. Z. **42**, 238–254 (1937); H. Görtler, Deutsche Math. **2**, 454–466 (1937); **3**, 189–200 (1938).] Let L_n denote the class of supportable domains for which the supporting function is a trigonometrical polynomial of degree n. A real number $\Delta_{n+1}(H)$ depending on the arbitrary supportable domain H is called a deficit with respect to L_n if $\Delta_{n+1}(H)=0$ implies $H \in L_n$. The isoperimetric deficit $L^2-4\pi F$, where the length L and the area F of H are defined by the usual formulae, is a deficit with respect to the class L_1 of circles. By means of the Fourier series of $H(\varphi)$ a sequence of deficits $\Delta_{n+1}(H)$ can easily be found. The author deals with the problem of finding such sequences which are calculable without knowledge of the Fourier development of $H(\varphi)$. This is solved in terms of convolutions of H with the powers of a "Polynombereichsfolge" T, i.e., an arbitrary but fixed trigonometric series. Several special cases which admit simple geometrical interpretations are discussed.

W. Fenchel (Princeton, N. J.).

Rosenbaum, R. A. Some characteristic properties of the circle. Math. Gaz. **33**, 273–275 (1949).

Let C be a smooth closed convex curve. A point P is called a π-point if the product of the segments into which a chord through P is divided by P is independent of the chord. A point P is called an α-point if, for every chord through P, the two angles, on the same side of the chord and containing the corresponding arc of C, between the chord and the tangents to C at its end points are equal. A point P is called a β-point if the bisecting normals of the chords through P are concurrent. An interior point of C which is a π-point is an α-point and conversely. The author proves several theorems stating that the existence of π-, α-, β-points implies that C is a circle. Examples: If there is an α-point on C then C is a circle; if there are two β-points inside C then C is a circle. Generalizations and related problems are briefly discussed.

W. Fenchel (Princeton, N. J.).

Biernacki, Mieczysław. Sur les cercles et sur les sphères qui passent par 3 ou 4 points d'un continu. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 3, 85–102 (1949). (French. Polish summary)

Consider the upper bound U and the lower bound u of the curvatures of circles containing at least three points of a continuum in Euclidean 3-space. Similarly let W and w be the upper and lower bounds of the curvatures of spheres having at least four points in common with a continuum. The primary aim of the paper is to contribute to the solution of the problem as to when these bounds can be attained. It is shown that the finiteness of U or W imposes very heavy conditions on the continuum, and that in general the bounds are not attained by circles or spheres which still meet the continuum in the indicated number of points. A typical theorem is the following. If the bound U relative to a bounded continuum is finite, this continuum is a simple rectifiable Jordan curve having, for each parameter value, a directed tangent which varies continuously with the parameter and with a curvature defined almost everywhere. The bound U cannot be attained by a circle meeting the curve in three distinct points unless the curve contains a circular arc of curvature U, in which case the bound is assumed by a circle tangent to the curve.

S. B. Jackson (College Park, Md.).

Bakhvalov, S. Geodesics on the cube. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 127–139 (1946). (Russian. English summary)

If a closed geodesic on the surface of a cube is developed along a straight line segment l in a plane, together with the faces containing the geodesic, there results a connected configuration consisting of s horizontal rows of squares, the first (leftmost) square of the ith row lying above the last square of the $(i-1)$th row $(2 \leqq i \leqq s)$, with the endpoints of l at homologous points of the lower edge of the first square of the first row and the upper edge of the last square of the sth row. Conversely, any such configuration which contains a line segment l with endpoints as above, and which has the property that the lower edge of the first square of the first row and the upper edge of the last square of the last row represent the same edge of the cube, gives rise to a geodesic (namely, the path on the cube which develops onto l).

Let the number of squares in the ith row be t_i. Then either all t_i are the same, or m of the t_i equal some number a and $n=s-m$ of the t_i equal $a-1$. If $m=n$ then the rows have alternately $a-1$ rows and a rows; if $m>n$ then every row of $a-1$ squares is followed by a number of rows of a squares, so that if we arrange to begin with a row of $a-1$ squares we then have r_1 rows of a squares each, then one row of $a-1$ squares, then r_2 rows of a squares each, $\cdots$, then r_n rows of a squares each (each $r_i>0$, $r_1+\cdots+r_n=m$); if $m<n$ the situation is similar but with the roles of $a-1$ and a interchanged. Suppose $m>n$; then either m/n is an integer and every $r_i=m/n$, or x of the r_i equal a certain number $r_{\max}$ and $y-n-x$ of the r_i equal $r_{\min}=r_{\max}-1$. If $x=y$ then r_i are alternately $r_{\min}$ and $r_{\max}$; if $x>y$ then every r_i which equals $r_{\min}$ is followed by a number of r_i which equal $r_{\max}$, so that if we arrange so that $r_1=r_{\min}$ we have one $r_{\min}$, then λ_1 numbers $r_{\max}$, then one $r_{\min}$, then λ_2 numbers $r_{\max}$, $\cdots$, then λ_y numbers $r_{\max}$ (each $\lambda_i>0$, $\lambda_1+\cdots+\lambda_y=x$); if $x<y$ then the situation is similar but with the roles of $r_{\min}$, $r_{\max}$ interchanged. Suppose $x>y$; then either x/y is an integer and every $\lambda_i=x/y$, or u of the λ_i equal a certain number $\lambda_{\max}$ and $v=y-u$ of the λ_i equal $\lambda_{\min}=\lambda_{\max}-1$. If $u=v$ then the λ_i are alternately $\lambda_{\min}$ and $\lambda_{\max}$; if $u>v$ then every $\lambda_{\min}$ is followed by a number of $\lambda_{\max}$'s; if $u<v$ then every $\lambda_{\max}$ is followed by a number of $\lambda_{\min}$'s, etc. Furthermore we have (assuming $m>n$, $x>y$, $u>v$, $\cdots$) $m/n=r_{\min}+x/n$, $x/y=\lambda_{\min}+u/y$, $u/v=\cdots$ (if one of these inequalities is reversed we obtain a similar formula but with max instead of min). Thus if $m=11$, $n=4$ then $m/n=11/4=2+3/4$ so that $r_{\min}=2$, $r_{\max}=3$, $x=3$, $y=1$ and, since $x/y=3=$integer, $\lambda_{\min}=\lambda_{\max}=3$. The paper contains several additional observations. The English summary is completely inadequate.

E. R. Kolchin (New York, N. Y.).

Rezek, Joseph. A contribution to embracing the basic conceptions of the integral geometry within the scope of ideas of Lie's group theory. Časopis Pěst. Mat. Fys. **75**, 16–26 (1950). (English. Czech summary)

The paper is a posthumous publication of the 1938 doctoral thesis of the author. It is concerned with the integral geometry in a homogeneous space under a Lie group. A condition is given, in terms of the infinitesimal transformations, for the existence and unicity of a density function defined up to a constant factor. The case of two homogeneous spaces with the same group is discussed, and a process is given to determine the number of functionally independent simultaneous invariants. The author concludes by considering the examples of the Euclidean plane and the Euclidean space.

S. Chern (Chicago, Ill.).

Algebraic Geometry

***Hasse, Helmut. Punti razionali sopra curve algebriche a congruenze.** Reale Accademia d'Italia, Fondazione Allessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 85–140, Rome, 1943.

This is an extensive report on the author's work on the Riemann hypothesis for elliptic fields. Extensive use is made of Deuring's presentation of the theory of algebraic correspondences. The background for Abelian functions is developed, though the restriction to the elliptic case is forced because of the lack of a general intersection theory. The author's aim is stated thus. "Pubblico questo lavoro in lingua italiana per facilitare alla scuola algebrico-geometrica italiana l'accesso ad un campo, coltivato in Germania da diverse parti, in cui la geometria algebrica e l'aritmetica si danno la mano."

O. F. G. Schilling (Chicago, Ill.).

Caccioppoli, Renato. Residui di integrali doppi e intersezioni di curve analitiche. Ann. Mat. Pura Appl. (4) **29**, 1–14 (1949).

If $f(x, y)$, $g(x, y)$ are analytic functions of two complex variables in a domain D, A and B their zero varieties, and S a two-cycle in D not intersecting either of them, then, as the author demonstrates, the integral

$$(2\pi)^{-2}\int\int_S (fg)^{-1}(\partial(f, g)/\partial(x, y))dxdy$$

is a (positive or negative) integer giving a certain intersection number for the varieties A, B relative to S in D; and by computing such an integral in two different ways the author has a new scheme of proof for the theorem of Bézout that the number of intersections of two homogeneous polynomials is mn.

S. Bochner (Princeton, N. J.).

Sz.-Nagy, Gyula. Ein anschaulicher Beweis der ersten Plückerschen Formel. Publ. Math. Debrecen 1, 71–72 (1949).

Another proof of the Plücker formula, $k=n(n-1)-2d-3r$, for the class k of a plane algebraic curve of order n with d nodes and r cusps is given. The proof utilizes the so-called symmetrizing process of Steiner [W. Blaschke, Kreis und Kugel, Veit, Leipzig, 1916] and intuitive continuity arguments. *H. T. Muhly* (Iowa City, Iowa).

Martan, Fr. On the planar points upon the cubic surfaces having double points. Acta Fac. Nat. Univ. Carol., Prague no. 187 (1948), 17–25 (1948). (Czech and English)

Franchetta, Alfredo. Sulla superficie delle coppie non ordinate di punti di una curva algebrica, a moduli generali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 7, 327–367 (1948).

Let C be an algebraic curve of genus p with general moduli; and denote by F the surface representing the non-ordered pairs of points of C. If C, varying continuously, tends to a rational curve C_0, the surface F tends to a rational surface F_0, which represents the nonordered pairs of points of C_0. The aim of the author is to determine the systems of curves on F_0, which are limits of systems of curves on the variable surface F. This can be done by using a plane mapping of F_0; it is thus possible to reduce the study of the curves on F to the study of convenient systems of plane curves. The author determines completely the systems of plane curves, which are limits of systems of curves on F, subject to the condition that they be representatives of symmetric correspondences on C of a sufficient high index. As an application of his method, the author finds the number of symmetric correspondences of valency zero on C, determined by a convenient number of pairs of points on C. Besides, the author gives an algebraic-geometric proof of the known property that every correspondence on a curve of genus p with general moduli has a valency.

F. Conforto (Rome).

Jongmans, F. Les séries abéliennes sur une courbe algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 892–905 (1948).

Jongmans, F. Quelques propriétés nouvelles des séries abéliennes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 245–257 (1949).

Sia C_p una curva algebrica di genere p ($\geqq 2$), sulla quale esista un sistema regolare di q ($0<q<p$) integrali di prima specie riducibili $u_1, \cdots, u_q$. L'autore si propone di approfondire lo studio, gia iniziato da F. Severi [Atti Accad. Italia, Mem. Cl. Sci. Fis. Mat. Nat. 12, 337–430 (1942); questi Rev. 8, 222] delle serie abeliane γ_n d'ordine n e specie q, ossia delle totalità (algebriche) costituite dai gruppi di n punti della C_p, per i quali sia $\sum_{i=1}^{n} u_j(P_i) \equiv c_j$ (modd. periodi) con le c_j ($j=1, \cdots, q$) costanti. Ogni γ_n è una totalità (algebrica) di serie lineari complete, in quanto, se un gruppo di punti appartiene alla γ_n, appartiene alla γ_n tutta la serie lineare completa individuata da tale gruppo di punti. L'autore prova che le γ_n si possono dividere in "proprie" ed "improprie" a seconda che, essendo d la dimensione della γ_n come totalità di serie lineari complete, sia $d=p-q$ ovvero $d<p-q$. Nel primo caso, le serie lineari complete costitutenti la γ_n sono in corrispondenza biunivoca con i punti della varietà abeliana collegata ad un sistema di $p-q$ integrali riducibili, complementare del sistema $u_1, \cdots, u_q$; nel secondo caso, che può presentarsi solo per $n<p$, si ha anche $d \geqq n-q$ e $d \leqq n-1$. La serie abeliana generica γ_n, quando sia $n<p$, è sempre impropria e per essa è $d=n-q$. L'autore indica però esempi di curve C_p su cui esistono γ_n con $n<p$ per cui $d>n-q$, in particolare $d=p-q$. Dopo ciò, l'autore considera le serie abeliane speciali, cioè proprie e costituite da serie lineari speciali, dimostrando tra l'altro che simili serie (d'ordine $n \geqq p$) si presentano allora ed allora soltanto che esistano serie abeliane proprie d'ordine $<p-1$. L'autore generalizza poi il teorema di Riemann-Roch per le serie abeliane complete (considerate ora come varietà di gruppi di punti) dal caso delle serie proprie (gia trattato da F. Severi) al caso delle serie improprie. Segue qualche applicazione. Circa le γ_n proprie e complete, entro le quali la serie lineare generica di γ_n sia priva di punti fissi e composta con un'involuzione j_m, l'autore fa vedere che esse si dividono in due categorie: (a) j_m è lineare e variabile in una serie abeliana propria di specie q; l'esistenza di tali γ_n non è ancora provata da esempi. Esempi effettivi si debbono cercare in modo che sia $p-q+3 \leqq m \leqq 1+p/2$ e $p \geqq 8$; (b) j_m è fissa e di genere $\geqq p-q$; m è limitato dal più piccolo dei due numeri $(p-1)/(p-q-1)$ e $n/(p-q+1)$. Se g_N^R è la serie lineare completa entro una γ_N speciale, vale la disuguaglianza $2R \leqq N-(p-q)-1$ che estende un noto teorema di Clifford. Vengono infine determinate le serie abeliane per cui l'ultima relazione vale col segno di uguaglianza, trovandosi che di simili serie esistono a priori due tipi, per uno dei quali l'esistenza rimane ipotetica. *F. Conforto* (Roma).

Berzolari, Luigi. Sopra una certa classe di poligoni inscrittibili in una curva algebrica. Ann. Mat. Pura Appl. (4) 28, 17–20 (1949).

L'auteur généralise la notion de polygone principal étudiée par Severi [Mem. Accad. Sci. Torino (2) 51, 81–114 (1902)]. On appelle polygone principal de $k$ème espèce d'une courbe algébrique C d'ordre n et de genre p de S_r, un polygone inscrit $P_1P_2 \cdots P_\nu$ tel que le S_{k-1} osculateur en P_i soit incident au S_{r-k} osculateur en P_{i+1} ($i=1, \cdots, \nu$ et $P_{\nu+1} \equiv P_1$). La relation entre les extrémités d'un coté du polygone induit sur C une correspondance $T(\alpha, \beta)$ dont l'auteur détermine les caractères α, β et la valence en fonction de n, p et des rangs de C, qui peut avoir des rebroussements et des points stationnaires de toutes espèces. De ce résultat, l'auteur déduit le nombre des polygones principaux de $k$ème espèce ayant un nombre donné de sommets, puis, comme application, le nombre des cordes principales de $k$ème espèce ($\nu=2$).

L. Gauthier (Nancy).

Berzolari, Luigi. Sulle coniche osculanti delle curve razionali normali. Ann. Mat. Pura Appl. (4) 28, 13–15 (1949).

Soit C^n la courbe rationnelle normale de S_n. Le plan osculateur en un point P de C^n est coupé par les S_{n-2} osculateurs, en des points dont le lieu est la conique osculante en P. L'auteur démontre par deux méthodes indépendantes le théorème suivant. Les coniques osculantes à C^n décrivent une surface Φ d'ordre $2\mu(n-2)$ où $\mu=1$ si $n=4$ et $\mu=2$ si $n \neq 4$, à sections hyperplanes de genre $n-2-2/\mu$, sur laquelle elles forment un système Σ d'indice $2/\mu$. Le lieu des pôles d'un hyperplan générique par rapport aux coniques de Σ est une courbe rationnelle d'ordre $2(n-2)$. Dans Σ il existe $2(n-2)$ coniques tangentes à un hyperplan générique, et il y en a $2\mu(n-2)$ qui rencontrent un S_{n-2} générique. Ce résultat généralise celui obtenu pour $n=3$ par Cremona sur les cubiques gauches [mêmes Ann. (1) 1, 164–174, 278–295 (1858)]. *L. Gauthier* (Nancy).

Babbage, D. W. The H-quadric of $n+2$ points in $[n]$. J. London Math. Soc. **25**, 51–54 (1950).

If P_i, $i=0, 1, \cdots, n+1$, are $n+2$ general points in $[n]$, the locus of the sets of $2n$ Hessian points of P_i on the different rational normal n-ic curves through P_i is a quadric primal. *M. Piazzolla-Beloch* (Ferrara).

Büke, Macit. Les surfaces d'ordre n de del Pezzo dans l'espace projectif à n dimensions. II. Rev. Fac. Sci. Univ. Istanbul (A) **14**, 165–205 (1949).

Büke, Macit. Les surfaces d'ordre n de del Pezzo dans l'espace projectif à n dimensions. III. Rev. Fac. Sci. Univ. Istanbul (A) **15**, 1–46 (1 plate) (1950).

Besides the author's original paper on nonsingular Segre quartic surfaces [same Rev. (A) **12**, 80–106, 164–189, 255–288 (1947); these Rev. **9**, 56, 609], two chapters of the present memoir on the del Pezzo surfaces F^n have already appeared [same Rev. (A) **14**, 143–164 (1949); these Rev. **11**, 457]. The first chapter dealt with the nonsingular F^3 and the second with the nonsingular F^n for $n=6, 7, 8$, and 9. The papers under review are the third and final chapter, which is a comprehensive study and classification of all the singular del Pezzo F^n of the real domain ($3 \leqq n \leqq 8$). The backbone of such a classification, the knowledge of the singular F^n in the complex domain, is provided in a paper by G. Timms [Proc. Roy. Soc. London. Ser. A. **119**, 213–248 (1928)], and Timms refers to the fact that Schläfli had originally sub-classified the 20 types of singular F^3 according to the reality of the lines on these surfaces [Philos. Trans. Roy. Soc. London **153**, 193–241 (1864)]. Much of the present work, apart from the complication of the real domain, follows broadly on the lines of Timm's work, but there are some new features, besides reality, which are worthy of note. The classification starts, naturally, with the two nodal $F^{6,8}$ mapped respectively on a visible and on an invisible quadric cone K by means of the quadric sections of K. There is only one nodal F^7, obtained by projecting the visible $F^{6,8}$ from a point of itself; and so on. Generally, the information given about each F^n ($n<8$) is (a) its representation on a quadric cone or surface, sometimes both, in the real domain, (b) the configuration of lines on it, and other details, and (c) its derivation by projection from a nonsingular F^n. For each F^4 and each F^3 there is also given a description of the degeneration of the surface into a tetrad or triad of planes, with the limiting positions of the node or nodes and of the lines on the surface duly exhibited. For the F^4, these degenerations are a development of the author's earlier work on the degeneration of nonsingular Segre quartic surfaces [loc. cit.]; and for the F^3 they are an extension (which the reviewer did not find easy to follow) of the work of B. Segre on the degeneration of the nonsingular F^3 [The Nonsingular Cubic Surfaces, Oxford, 1942; these Rev. **4**, 254]. The memoir is amply provided with diagrams (there are 132 in all, of which this final chapter contains 113), and at the end there is appended a large chart showing all the F^n and indicating diagrammatically all the relations of projection described in the text. *J. G. Semple* (London).

Giambelli, Giovanni. I teoremi di Staudt e di Steiner per le ipersuperficie algebriche. Atti Accad. Gioenia Catania (6) **6** (1943–49), no. 10, 4 pp. (1950).

The well-known characterization of a hyperquadric V^2_{n-1} of the projective space S_n as the locus of self-conjugate points of a polarity defined by a symmetric bilinear form is extended to the hypersurfaces V^m_{n-1} of order m in S_n by considering multilinear correspondences defined by multilinear forms in m sets of $n+1$ independent variables. An analogue of the Steiner definition of a conic is also given. *H. T. Muhly* (Iowa City, Iowa).

Vaona, Guido. Curve e superficie quasi-asintotiche della varietà di Grassmann che rappresenta le rette di uno spazio lineare. Boll. Un. Mat. Ital. (3) **4**, 360–367 (1949).

The author studies the quasi-asymptotic curves and surfaces of the Grassmann variety W_t representing the straight lines of a projective space S_n. The definition of quasi-asymptotic variety V_h imbedded in a variety V_k of a projective space is due to Bompiani [Proc. 5th Internat. Congress Math. 1912, vol. 2, pp. 22–27 (1913)] and is as follows. A V_h is said to be a quasi-asymptotic variety σ^q_{rs} (γ_{rs} for a curve) when the osculating space $S(r)$ to V_h at the generic point of V_h and the osculating $S(s)$ to V_h at the same point are conjuncted by a linear space of less dimension than ordinary. In this paper the following results are stated. (a) The quasi-asymptotic γ_{12} of W_4 consists of the curves which are images of developable ruled surfaces (or, in particular, of cones and plane ruled surfaces; (b) the γ_{12} (not γ_{11}) of W_t consists of the curves which represent ruled surfaces belonging to the S_3's of S_n. The author finds then that any Grassmann variety has quasi-asymptotic surfaces σ^q_{12} of three different kinds, depending on arbitrary functions of two variables, namely: (i) the quasi-asymptotic σ^3_{12} of the 3rd kind (for which, i.e., the osculating $S(2)$ at a point P belongs to the S_t, tangent to W_t at P; for the 2d and 1st kind $S(2)$ and S_t are conjuncted by spaces at $t+1$ and $t+2$ dimensions, respectively) represent, beside the ruled planes of S_n, the cones obtained by projecting from a point any surface; (ii) the σ^2_{12} of the 2d kind represent V_3's of S_n ($n>3$), loci of ∞^2 straight lines, having a fixed tangent S_3 along any line; (iii) the σ^1_{12} of the 1st kind represent V_3's of S_n ($n>3$), loci of ∞^2 straight lines, having a fixed tangent plane along each line, but variable tangent S_3. *V. Dalla Volta.*

Segre, Beniamino. Sulle V_n contenenti più di ∞^{n-k} S_k. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 193–197 (1948).

Segre, Beniamino. Sulle V_n contenenti più di ∞^{n-k} S_k. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 275–280 (1948).

The construction of all varieties V_n containing an ∞^r complete continuous system of S_k, $r=1, \cdots, n-k$, being obvious, the two following problems arise: (a) to determine for any n and k ($0<k<n$) the values of the dimension $d=d(n, k)$ of a complete continuous system of S_k filling a V_n, which can in fact be obtained on some V_n; (b) to classify the V_n's with ∞^d S_k, where $d>n-k$. Without loss of generality one may suppose the V_n to be filled up by the ∞^d S_k. To approach the solution of the questions (a) and (b) the author gives first some examples of V_n with an ∞^d complete continuous system of S_k, where $d>n-k$: linear projective spaces, specialized quadrics, and Grassmannian varieties. From these examples one may deduce in particular that, by putting $\delta(n, k)=(k+1)(n-k)$, $d(n, 1)$ may assume all the values between n and $\delta(n, 1)=2n-2$, while $d(n, 2)$ may assume all the values between $n-1$ and $\delta(n, 2)=3n-6$ with at most the exception of $d=3n-7$. After this, the author, aiming to give at least partial answers to the questions (a) and (b), proves some lemmas whose proofs are based on various suitable methods of counting constants. Then the following theorems are proved. It is always true that

$d(n, k) \leqq \delta(n, k)$; moreover, $d(n, k) = \delta(n, k)$ only for the linear projective spaces S_n. If $k > 1$ the values $\delta(n, k) - i$, $1 \leqq i \leqq k-1$, are lacunary ones for $d(n, k)$, i.e., they are never attained; and, moreover, if $d(n, k) = \delta(n, k) - k$, V_n is the locus of $\infty^1 S_{n-1}$. In addition, if $k > 2$, the values $\delta(n, k) - j$, $k+1 \leqq j \leqq 2k-2$, are lacunary ones. After this, only the case when $d(n, k) \leqq \delta(n, k) - 2k + 1$ remains to be investigated. The author proves that $d(n, k) = \delta(n, k) - 2k + 1$ characterizes certain specialized quadrics. The value $d(n, k) = \delta(n, k) - 2k$ is never a lacunary one, while the value $d(n, k) = \delta(n, k) - 2k - 1$ is lacunary for every k and $n \geqq k+3$, the cases $k = 1, 2, 4$ excepted. *F. Conforto* (Rome).

Segre, Beniamino. Alcune questioni diofantee. Boll. Un. Mat. Ital. (3) **5**, 33–43 (1950).

A geometrical proof is given of the theorem of Hurwitz [Mathematische Werke, vol. 2, pp. 446–468, in particular, § 8 = Vierteljschr. Naturforsch. Ges. Zürich **62**, 207–229 (1917)] that if a plane binary cubic curve of genus 1, with rational coefficients, has only four rational points then those points together with their tangents form one of two configurations. In one of these cases the cubic can be transformed to (1) $x(x+z)(y+z)+ay^2z=0$ in homogeneous coordinates x, y, z where a is rational [Hurwitz, loc. cit.]. The author uses a method of descent to prove that for $a = -1/12$ there are indeed only four rational points [similar results hold for $a = -1/4$ [Hurwitz, loc. cit.] and for $a = -1/8$ [B. Levi, Atti Accad. Sci. Torino **43**, 99–120 (1908)]]. This result is used to show that there are no nontrivial rational solutions of

$$(2) \qquad \begin{aligned} x_1^2+y_1^2=x_2^2+y_2^2=x_3^2+y_3^2;\\ x_1^3+y_1^3=x_2^3+y_2^3=x_3^3+y_3^3. \end{aligned}$$

The curve (2) (in homogeneous coordinates) decomposes into a curve of degree 8 with no real points, six quartics, and four lines. The quartics are equivalent to (1) with $a = -1/12$ and so all their rational solutions are known (and trivial); the lines give trivial solutions. This solves a problem of Moessner [same Boll. (3) **4**, 146 (1949)]. *J. W. S. Cassels* (Cambridge, England).

Hodge, W. V. D. On the topology of three-folds whose hyperplane sections have geometric genus zero. Ann. Mat. Pura Appl. (4) **29**, 115–119 (1949).

A p-cycle Γ_p on (the Riemannian of) an algebraic variety V_m is said to be embedded if it lies on a subvariety V_n' with $n < p$. It is proved that if an algebraic variety V_3, with only normal singularities in S_4, has the property that its prime sections are of geometric genus zero, then every 3-cycle on it is embedded. The result is proved by a combination of methods due to Lefschetz and the author, and some natural generalisations are given. *D. B. Scott* (London).

***de Franchis, Michele. Matrici di Riemann e varietà abeliane.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 43–84, Rome, 1943.

An expository survey of the subject, with a detailed bibliography. *J. A. Todd* (Cambridge, England).

Wylie, C. R., Jr. A new series of line involutions in S_3. Math. Mag. **23**, 125–131 (1950).

In this paper the author is concerned with involutorial line-transformations in S_3, which can be considered as involutorial point-transformations on a hyperquadric V_4^3 in S_5. The author considers the involutions for which the line joining a general point P to its image P' lies entirely on V_4^3. These involutions possess four important numerical characteristics: m, the order of the involution; i, the order of the complex of invariant elements; n, the order of the ruled surface formed by the lines which join the points of a generic line l to their respective images; k, the number of points of l with the property that the lines joining them to their images lie entirely on V_4^3. These four quantities are not independent, but satisfy the identity $m+1 = 2n-k = n+i$. In the present paper there is shown the existence of involutions associated with all possible sets of characteristics except those in which the lines left invariant form a linear complex. *M. Piazzolla-Beloch* (Ferrara).

Benedicty, Mario. Sul gruppo delle trasformazioni cremoniane monomiali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 697–702 (1949).

L'autore dimostra che il gruppo delle trasformazioni cremoniane monomiali è generato da un gruppo continuo di affinità e da tre sue trasformazioni particolari, due omografiche ed una quadratica. *M. Piazzolla-Beloch.*

Villa, Mario. Sul prodotto di due trasformazioni puntuali. Ann. Mat. Pura Appl. (4) **29**, 307–313 (1949).

It is known [e.g., Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 718–724 (1942); (7) **4**, 1–7 (1943); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 55–61, 192–196, 295–303 (1948); these Rev. **8**, 219; **10**, 210] that the neighbourhood of the 2d order of a pair of regular corresponding points in a point transformation between two projective planes is determined by three pairs of directions (characteristic directions), and by three projectivities, one between the directions of each pair. In this paper the author assumes as given two point transformations, T_1 (between a plane π and a plane π'), and T_2 (between a plane $\bar{\pi} \equiv \pi'$ and a plane π''); besides the characteristic directions and projectivities for a (regular) pair of points O, O', corresponding to each other in T_1, and the analogous elements for the points $\bar{O} \equiv O'$ and O'' (corresponding to each other in T_2) are given. The author shows how the characteristic direction for T, through O, may be determined by solving a certain cubic equation whose coefficients are formed by cross-ratios obtained by means of the characteristic directions of T_1 and T_2 through O'; also, the projectivity determined between the pencils O and O'' by T is the product of the analogous projectivities (between the pencils O, O', and between the pencils $\bar{O} \equiv O', O''$) determined, respectively, by T_1 and T_2. The characteristic directions for T through O'', and the characteristic projectivities for T, are also obtained. *V. Dalla Volta* (Princeton, N. J.).

Salini, Ugo. Trasformazioni razionali osculatrici ad una trasformazione puntuale fra due spazi ordinari in una coppia di punti ad Jacobiano nullo di caratteristica uno. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 1–27 (1949).

[From the author's paper.] The author continues an investigation begun in a preceding paper [same Rend. Mat. e Appl. (5) **7**, 44–71 (1948); these Rev. **10**, 266] concerning one-to-one point transformations $T(1)$ between two linear spaces S_3, S_3', in a pair of points (O, O'), where the Jacobian is zero and of rank one. In this paper certain rational transformations T_r are introduced, which approximate the given $T(1)$ up to the neighbourhood of the 3rd order. By a suitable choice of a particular T_r, T^*_r, it is possible to determine completely a projective coordinate system in the two spaces, which is intrinsically related to $T(1)$, to give a canonical

form to the equations of $T(1)$; moreover, the given transformation may be expressed, up to the 2d order, by means of four invariants for which a geometrical interpretation is given. *V. Dalla Volta* (Princeton, N. J.).

Salini, Ugo. Sulle trasformazioni puntuali fra due spazi ordinari in una coppia di punti ad Jacobiano nullo di caratteristica uno, quando la superficie Jacobiana presenta nel punto un punto doppio biplanare ed uno dei piani coincide col piano stazionario. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 229–245 (1949).

[From the author's paper.] This paper continues from another point of view the author's research [see the preceding review] and deals with the case when the tangent cone to the Jacobian surface at the point O, which is double for the Jacobian, is decomposed into two distinct planes. The research is extended up to the 2d order, and an intrinsic projective coordinate system in the two stars O, O' is given. However, contrary to what happened in the case of a double conic point, the coordinate system cannot be completely determined except by means of the neighbourhood of the 3d order. Moreover, there are no longer four characteristic lines, but only two, and this number is still further decreased in the case, to be studied in a future paper, of the uniplanar double point. The author introduces other geometrical objects with an intrinsic relation to the stars O, O', and the geometrical meanings of some projective invariants of the transformations are given. *V. Dalla Volta.*

Differential Geometry

Zaat, Jacobus. Differentialgeometrie der Betragflächen analytischer Funktionen. Mitt. Math. Sem. Univ. Giessen **30**, iii+38 pp. (1944).

Let $f(z)$ be analytic in some region R of the z-plane, $z=x+iy$; the three-dimensional surface whose coordinates are x, y, and $w=|f(z)|$ is called [e.g., Jahnke-Emde, Funktionentafeln . . . , Teubner, Leipzig-Berlin, 1933] the relief of the function $f(z)$ over R. The author calculates explicitly in terms of $f(z)$ the formulas of differential geometry for the relief of $f(z)$ and includes in his dissertation four sketches which do not appear in the table of functions cited above. *A. J. Lohwater* (Ann Arbor, Mich.).

Sibirani, Filippo. Alcuni tipi di rigate con assegnata linea di gola. Ann. Mat. Pura Appl. (4) **30**, 261–265 (1949).

Étant donnée une courbe C, l'auteur détermine les surfaces réglées admettant C pour ligne de striction, en se placant successivement dans les cas où la génératrice de la surface réglée fait un angle constant avec la tangente à C, la normale principale, la binormale, ou une direction quelconque donnée. Les résultats obtenus dans le premier et le troisième cas mettent en évidence le fait que si la génératrice issue d'un point quelconque P de C est située dans le plan rectifiant en P, elle est invariablement liée au trièdre de Frenet de sommet P. Toute droite invariablement liée au trièdre de Frenet engendrant une surface réglée admettant C pour ligne de striction est d'ailleurs nécessairement située dans le plan rectifiant. *P. Vincensini* (Marseille).

Hartman, Philip, and Wintner, Aurel. On the embedding problem in differential geometry. Amer. J. Math. **72**, 553–564 (1950).

The following imbedding theorem concerning surfaces in 3-dimensional Euclidean space is proved. Let $n\geqq 2$ and $0<\mu<\lambda<1$ and let $g_{ik}(u, v)$ be given functions (whose 2×2 matrix is symmetric and positive definite) having partial derivatives up to order n which satisfy a uniform Hölder condition of order λ with respect to all the variables. Let the Gaussian curvature, relative to the g_{ik}, at every point (u, v) in a neighborhood of $(0, 0)$ be zero or else have constant sign. Then there exist three functions $x^i=x^i(u, v)$ having partial derivatives of order n which satisfy a uniform Hölder condition of order μ and also satisfy $\sum_{j=1}^{3}(\partial x^j/\partial u^1)(\partial x^j/\partial u^k)=g_{ik}$ $(u^1=u,\ u^2=v)$ in a neighborhood of $(0, 0)$. The proof when the points (u, v) are all elliptic or hyperbolic depends upon a lemma concerning the existence of solutions of a certain Monge-Ampère differential equation. The proof of the parabolic case is more immediate. *A. Fialkow* (Brooklyn, N. Y.).

Backes, Fernand. Sur les cercles dont les dix coordonnées pentasphériques satisfont à une même équation de Laplace. C. R. Acad. Sci. Paris **231**, 106–108 (1950).

Dans une note récente [mêmes C. R. **230**, 1252–1253 (1950); ces Rev. **11**, 687] l'auteur a montré l'intéret qui s'attache, pour certaines questions géométriques, à la considération d'un pentasphère oblique mobile. Dans la note actuelle, il donne une application de la méthode du pentasphère oblique mobile à l'étude des congruences de cercles telles que les dix coordonnées du cercle générateur, qui possède des sphères focales, sont solutions d'une même équation de Laplace du type hyperbolique. Il montre que la congruence des axes des cercles d'une telle congruence est une congruence rectiligne, indique une réciproque, et signale une suite périodique spéciale de réseaux de Laplace de période quatre. *P. Vincensini* (Marseilles).

***Blaschke, Wilhelm. Ueber die Massbestimmungen von Hermite.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 391–408, Rome, 1943.

The first three sections are devoted to the algebraic properties of Staudt chains and normal chains and to the Hermite elliptic metric. In the remaining sections the following topics are briefly discussed: differential geometry of one, two, and three real dimensions and integral geometry, as well as the Kähler space. We describe here as an example the differential geometry of two real dimensions. Let x_0, x_1, x_2 be projective coordinates of a point $\mathfrak{x}$, normalized by (1a) $\mathfrak{x}\cdot\bar{\mathfrak{x}}=x_0\bar{x}_0+x_1\bar{x}_1+x_2\bar{x}_2=1$ and let $\bar{\mathfrak{x}}=\bar{\mathfrak{x}}(u^{\mathrm{I}}, u^{\mathrm{II}})$ be a function of two real parameters u^{ν} $(\nu, \lambda, \mu=\mathrm{I}, \mathrm{II})$. The function Γ_λ defined by (2) $\bar{\mathfrak{x}}\cdot\partial_\lambda\mathfrak{x}=i\Gamma_\lambda$ are real as follows from (1a). If we put (3) $\mathfrak{x}_{,\lambda}=\partial_\lambda\mathfrak{x}-i\Gamma_\lambda\mathfrak{x}$, then we see that the normalization (1a) admits the transformation (1b) $\mathfrak{x}^*=e^{i\sigma}\mathfrak{x}$, whereby $\mathfrak{x}^*_{,\lambda}=e^{i\sigma}\mathfrak{x}_{,\lambda}$. If $a_{\lambda\mu}+ik_{\lambda\mu}=\mathfrak{x}_{,\lambda}\cdot\bar{\mathfrak{x}}_{,\mu}$, then the tensors $a_{\lambda\mu}$ and $k_{\lambda\mu}$ are invariant under (1b) and moreover

$$a_{\lambda\mu}=a_{(\lambda\mu)}=\partial_{(\lambda}\mathfrak{x}\cdot\partial_{\mu)}\bar{\mathfrak{x}}-\Gamma_{(\lambda}\Gamma_{\mu)},$$

$$k_{\lambda\mu}=k_{[\lambda\mu]}=\frac{1}{i}\partial_{[\lambda}\mathfrak{x}\cdot\partial_{\mu]}\bar{\mathfrak{x}}=\partial_{[\mu}\Gamma_{\lambda]}.$$

Consequently,

$$A\equiv\int\int(a_{\mathrm{II}\,\mathrm{II}}a_{\mathrm{II}\,\mathrm{II}}-a^2_{\mathrm{I}\,\mathrm{II}})^{\frac{1}{2}}du^{\mathrm{I}}du^{\mathrm{II}},$$

$$B\equiv\int\int k_{\mathrm{I}\,\mathrm{II}}du^{\mathrm{I}}du^{\mathrm{II}}=\frac{1}{2}\oint\Gamma_\lambda du^\lambda=\frac{1}{2i}\oint\bar{\mathfrak{x}}\cdot d\mathfrak{x},$$

$$C\equiv\int\int(D\bar{D})^{\frac{1}{2}}du^{\mathrm{I}}du^{\mathrm{II}}$$

(where D is the determinant of the points $\bar{\mathfrak{x}}, \bar{\mathfrak{x}}_\lambda$) are integral invariants related by $(dA)^2-(dB)^2=(dC)^2$. Instead of $\mathfrak{x}, \mathfrak{x}_\lambda$

we may introduce another frame which does not depend on the choice of u^r, namely the frame consisting of $\mathbf{x}$, $\mathbf{p}$, and ${}^*\mathbf{p}$ where $\mathbf{p}=v^\lambda\mathbf{x}_\lambda$, ${}^*\mathbf{p}=\bar{v}^\lambda\mathbf{x}_\lambda$ are two mutually perpendicular points $\mathbf{p}\cdot{}^*\mathbf{p}\equiv v^\lambda\bar{v}^\mu a_{\lambda\mu}=0$ on the "polar line" of $\mathbf{x}$.

V. Hlavatý (Bloomington, Ind.).

Buscheguennce, S. S. Sur un cas des congruences stratifiables. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 140–149 (1946). (Russian. French summary)

A projective generalization of Bianchi's stratifiable pair of congruences whose corresponding lines l_1 and l_2 are orthogonal, their common perpendiculars passing through a fixed point. These three lines meet the plane at infinity in three points which form a self polar triangle with respect to an imaginary circle in the plane at infinity. The generalization is obtained by taking a fixed point Ω and a fixed plane ω and a line of the second order c^2 in this plane. Let M_1, M_2 be the points in which l_1 and l_2 meet ω, and M the point in which the line through Ω intersecting l_1 and l_2 meets ω. The paper investigates the properties of stratifiable congruences for which the triangle MM_1M_2 is self polar with respect to c^2 and in particular for a degenerate c^2. In general, the differential equations admit solutions which depend on four functions of one parameter. *M. S. Knebelman.*

Rosenfeld, B. m-dimensional generalisation of Clifford's surface. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 150–154 (1946). (Russian. English summary)

Clifford's surface is generalized to n dimensions by considering a $(2n-1)$-dimensional elliptic space $R^{(2n-1)}$ (of curvature 1) and regarding it as a sphere in a $2n$-dimensional Euclidean space, identifying diametrically opposite points. In this Euclidean space an orthonormal vector basis may be chosen whose extremities are on $R^{(2n-1)}$, such that if $e_1, \cdots, e_{2n}$ are the vectors then e_i and e_{n+i} are the nonintersecting edges of a self-polar polyhedral. Any point in $R^{(2n-1)}$ then has coordinates $x=\sum(e_i\cos\lambda_i+e_{n+i}\sin\lambda_i)\cos\rho_i$, where $0\leqq\rho_i<\pi$, $\sum\cos^2\rho_i=1$ and $0\leqq\lambda_i\leqq\pi/2$. By letting ρ_i be constant we get a surface $K^{(n)}$ whose equations are

$$x_i=\cos\lambda_i\cos\rho_i,\quad x_{n+i}=\sin\lambda_i\cos\rho_i,$$

so that $ds^2=\sum\cos^2\rho_i d\lambda_i^2$. This is a Euclidean metric and the equations of $K^{(n)}$ may be written as $x_i^2+x_{n+i}^2=\cos^2\rho_i$, which is a generalization of Clifford's equations ($n=2$).

M. S. Knebelman (Pullman, Wash.).

Engel, Friedrich. Die Translationsflächen im gewöhnlichen Raume. Mitt. Math. Sem. Univ. Giessen **36**, vi+117 pp. (1945).

This monograph is divided into two parts. The first part consists of six sections. In the first two sections the Lie equation,

(1a) $$D\equiv r+(u(p,q)+v(p,q))s+u(p,q)v(p,q)t=0,$$

for a surface of translation is derived, as well as the Scheffer's condition

(1b) $$u_q=uu_p,\quad v_q=vv_p,$$

by means of geometrical considerations based on the loci of ideal points of the tangents to the curves of translation. In the next section the integral surfaces of (1) (not only the surfaces of translation) are found in two different ways, where the second is based solely on infinitesimal transformations whose trajectories are the curves of translations. All possible special cases are considered. Section four is devoted to the case $u=v$, which does not lead in general to a surface of translation. In the fifth section the integral surfaces of (1) and of

(1a)* $$D^*\equiv r+(u^*(p,q)+v^*(p,q))s+u^*(p,q)v^*(p,q)t=0,$$

(1b)* $$u_q{}^*=u^*u_p{}^*,\quad v_q{}^*=v^*v_p{}^*$$

are treated. Finally, in the sixth section the author solves the problem of finding under what conditions any equation of the pencil $D+\lambda D^*=0$ is of the type (1), and, moreover, the corresponding integral surfaces are considered.

The second part consists of sections 7–10. The considerations of section 7 are based on a differential equation

(2a) $$r+(u(x,y)+v(x,y))s+u(x,y)v(x,y)t=0$$

of surfaces of translation whose sets of curves of translations always have the same projections on the (x,y)-plane. Even here conditions similar to (1b) are valid:

(2b) $$u_x+vu_y=0,\quad v_x+uv_y=0.$$

The equations (2) admit for solution only surfaces of translation. The comparison of (1) and (2) leads to a classification of surfaces of translation (section 8). The chief result of section 9 may be stated as follows: A necessary and sufficient condition that the equation $z=\varphi(x,y)$ represent a surface of translation is that φ satisfy a certain system of two partial differential equations of order 5. In the last section a problem similar to that of section 5 is treated; it is based on equations of type (2). *V. Hlavatý.*

Penzov, Yu. E. On differential-geometrical objects of class v in X_1. Mat. Sbornik N.S. **26(68)**, 161–182 (1950). (Russian)

A detailed study by means of group-theoretic considerations of the possible differential geometric objects in X_1. With each geometric object of class v one can associate two groups $\mathfrak{A}^{(1,v)}$ and $\mathfrak{B}^{(1,v)}$ which are shown to be the first and second parametric groups of the Lie group of all derivatives of orders up to and including v. They are simply isomorphic and, with the proper choice of generators, their structure constants are given by $c^r_{st}=(t-s)\delta^r_{s+t-1}$. By means of this group and a number of theorems on transitive geometric objects the author is enabled to give a complete classification of such objects with either 1 or 2 components. Those of one component are particularly interesting; they are $'\Omega=\Omega/f'$ (a density); $'\Omega=\Omega/f'-f''/f'^2$ (affine connection) and $'\Omega=\Omega/f'^2+\frac{1}{2}f''^2/f'^4-f'''/f'^3$ (a projective connection). There are 7 geometric transitive objects of 2 components in X_1. Among the theorems proved is an arithmetic one; if $N\geqq3$, $2\leqq h\leqq N-1$, then there exists at least one pair of integers x and y, $1\leqq x<y\leqq N-1$, for which

$$f(x,y)=x+y+h+1+\sum_{i=1}^{x}\sigma_i+\sum_{j=1}^{y}\sigma_j-2N=0$$

(all letters stand for nonnegative integers) and

$$\sum_{i=1}^{N-1}\sigma_i\leqq N-h-1.$$

M. S. Knebelman (Pullman, Wash.).

Cossu, Aldo. Sulle varietà a due dimensioni subordinate ad una varietà a tre dimensioni dotata d'una connessione affine. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 186–199 (1949).

It is known [cf., e.g., E. Bortolotti, Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (2) **64**, 441–463 (1931); the

author makes several references to this paper] that, given an A_3 with a nonsymmetric affine connection ∇ of parameters $\Gamma^{\nu}_{\lambda\mu}$, $\lambda, \mu, \nu = 1, 2, 3$, and an A_2 imbedded in A_3, if we consider, for any point of A_2, a pseudonormal direction C' (not lying in the tangent plane to A_2 at the considered point), then the connection ∇ induces two linear connections, the former, ∇' on the anholonomic variety of the pseudonormal directions, the other, ∇'' on A_2. Besides, the reviewer has proved [same Rend. Mat. e Appl. (5) **5**, 271–282, 315–326 (1946); these Rev. **9**, 159] that any 2-plane in a point P of A_2 and its direction of torsion (due to the skew-symmetric part of $\Gamma^{\nu}_{\lambda\mu}$) correspond to each other in a reciprocity (in the star with center in P); consequently, for any point P, there is a quadric cone Γ (with vertex P) enveloped by the 2-plane containing its direction of torsion; these directions, in turn, belong to another quadric cone C (also with vertex P). With these premises, the author gives many results relative to an A_2 imbedded in an A_3 and to the connection induced on A_2 by a choice of a field of pseudonormal directions C'. The author begins by geometrically characterizing the A_2 which in any point touch the corresponding cone Γ; this characterization is independent of the choice of C', and involves the symmetric connections [Bortolotti, Ann. Mat. Pura Appl. (4) **8**, 53–101 (1930)] attached to ∇, ∇', and ∇''. After giving some results for particular positions of the cones Γ and C, the author proves that for any A_2 not tangent to Γ, a choice of the pseudonormals is possible for which the induced connection is symmetric. Other results of the paper concern some particular cases of the above-mentioned correlation between 2-planes and their directions of torsion. *V. Dalla Volta* (Princeton, N. J.).

Alda, Václav. Sur les propriétés affines des correspondances analytiques. Časopis Pěst. Mat. Fys. **75**, 51–67 (1950). (French. Czech summary)

Let C be an analytic correspondence (not an affine transformation) between the points A and B of two affine spaces (A) and (B) of n dimensions such that $B = CA$. There exists an affine transformation T such that $TA = B$, $TdA = dB$. The author calls this transformation the tangent affinity to the correspondence C at the point A. For second order differentials the inequality $Td^2A \neq d^2B$ generally holds. Let K denote a curve passing through A. Because of the above inequality the curves K and $T^{-1}CK$ usually make contact of only the first order. If these curves are projected in the direction $V = T^{-1}d^2B - d^2A$ onto a hyperplane through A, the hyperplane sections have second order contact at A. In the case of a plane ($n = 2$) there exists an affinity H such that $HA = B$, $HdA = dB$, $Hd^2A = d^2B$, so that V is a direction such that the curves CK and THK have contact of second order.

A characteristic direction of the correspondence C is a direction such that to each curve passing through A in the direction and having an inflexion at A corresponds a curve having an inflexion at B. A curve in the space (A) or (B) whose direction at a generic point is a characteristic direction is called a characteristic curve. The correspondences C between two planes are investigated for the case in which the Jacobian of the tangent affinity T is constant. Such correspondences exist and depend on one function of two independent variables. For the cases in which there exist (for a correspondence C) at least a two fold characteristic direction the equations or constructions of such correspondences are given. A characteristic curve corresponding to a two fold characteristic direction is always a straight line, and the solutions fall into four classifications, namely, the characteristic lines of (A) and (B) (a) are parallel in (A) (and in (B)), (b) form pencils of lines in (A) and (B), (c) form a pencil of lines in one plane (A) (or (B)) and envelop a curve in the other plane, (d) envelop curves in both planes. The correspondences for plane and space are sought ($n = 2$ or 3), for which the affinities T depend on $\nu < n$ parameters. In this case, there passes through each point A an $(n-\nu)$-dimensional linear space $R_{n,\nu}$ on which T is constant. Furthermore, for this space the correspondence C is identical with the affinity T. To it corresponds in space (B) a linear space $R'_{n,\nu}$ passing through the point $B = CA$. The correspondence of the type for which $n = 2$, $\nu = 1$, depends on three functions of one variable. The equations of the correspondences for the cases in which the straight lines $R_{2,1}$ are parallel or form a pencil are given. In case the straight lines $R_{2,1}$ envelop a curve, the corresponding lines $R'_{2,1}$ also envelop a curve. The correspondences of the types (B) and (C) for which $n = 3$, $\nu = 1$, and $n = 3$, $\nu = 2$, respectively, are studied. Correspondences of type (B) depend upon five functions of one variable; those of type (C) depend upon five functions of two variables. Constructions for the various cases of these types are given. The spaces $R_{2,2}$ for a case of type (C) are the generators of a congruence.

P. O. Bell (Lawrence, Kans.).

★Vranceanu, G. Proprietà topologiche differenziali delle varietà anolonome. Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 507–523, Rome, 1943.

This is a report on problems connected with the theory of congruences of curves on an X_n^m. The first five sections deal with the more or less elementary questions of this field. Example: Let $n - \sigma$ be the least number of differentials which may appear in the Pfaff system defining an X_n^m. Then this system may be written $\lambda_j^{\alpha} dx^j = 0$, $\alpha = m+1, \cdots, n$, $j = \sigma+1, \cdots, n$, and admits the solutions $x^j = c^j$. Hence σ of the congruences may be taken as consisting of lines parallel to the coordinate axis $x^1, \cdots, x^{\sigma}$. In some cases the remaining $m - \sigma$ congruences may also consist of straight lines. The last three sections deal briefly with questions connected with the Cartan equivalence problem $ds^{i'} = c_i^{i'} ds^i$, $i = 1, \cdots, n$. Example: Let X_n^m be given by $ds^h = 0$, $h = m+1, \cdots, n$. These forms may be completed to a system of n independent forms known up to the transformations

$$(1) \qquad \begin{aligned} ds^{\alpha'} &= c_{\alpha}^{\alpha'} ds^{\alpha} + c_h^{\alpha'} ds^h \qquad (\alpha = 1, \cdots, m), \\ ds^{h'} &= c_h^{h'} ds^h \qquad (h = m+1, \cdots, n), \end{aligned}$$

Hence we have for the operators $X_i = \partial/\partial s^i$,

$$(2) \quad (a) \quad X_{\alpha} = c_{\alpha}^{\alpha'} X_{\alpha'}, \qquad (b) \quad X_h = c_h^{\alpha'} X_{\alpha'} + c_h^{h'} X_{h'}.$$

If this group has to reproduce the congruences given by $X_1 f, \cdots, X_m f$, then (2) (a) must reduce to

$$(2) \qquad (c) \quad X_1 = c_1^{1'} X_{1'}, \cdots, X_m = c_m^{m'} X_{m'}$$

and consequently (1) simplifies to

$$(3) \qquad \begin{aligned} ds^{1'} &= c_1^{1'} ds^1 + c_h^{1'} ds^h \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \\ ds^{m'} &= c_m^{m'} ds^m + c_h^{m'} ds^h \\ ds^{h'} &= c_h^{h'} ds^h. \end{aligned}$$

The group (3) preserves a system S of equations $ds^h = 0$ and m (noncomplementary) systems S_{α} which one obtains if one associates S with $ds^{\alpha} = 0$. *V. Hlavatý.*

Norden, A. P. On normalized surfaces of a conformal space. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 105–122 (1950). (Russian)

This is a continuation of the author's researches in conformal differential geometry over the past fifteen years. It is a detailed study of normalized surfaces in a three-dimensional conformal Möbius space. With each normalization is associated a Weyl metric and an inner affine connection. Their variation under changes of normalization are investigated and the integrability conditions are given a geometric interpretation. In the author's words: A detailed examination of various possible normalizations brings into order the results in conformal theory of surfaces with the geometry of surfaces of constant curvature in Euclidean three-space. *M. S. Knebelman* (Pullman, Wash.).

Lichnerowicz, André. Sur certaines classes d'espaces riemanniens compacts. C. R. Acad. Sci. Paris **230**, 2146–2148 (1950).

The following theorems are stated for a compact Riemannian manifold V_n with positive definite metric. (1) Let $Q=H_{\lambda\alpha\beta\gamma}{}^{\lambda}{}_{\mu}R^{\mu\alpha,\beta\gamma}$, where $H_{\alpha\beta\gamma\delta,\lambda\mu}$ is defined by E. Cartan [Leçons sur la géométrie des espaces de Riemann, 2d ed., Gauthier-Villars, Paris, 1946, p. 265; these Rev. **8**, 602]. Then if Q is nonnegative and (a) if $\nabla_\alpha R_{\beta\gamma}=0$, V_n is symmetric; (b) if $R_{\beta\gamma}=0$, V_n is locally Euclidean. (2) Every recurrent V_n [H. S. Ruse, Proc. London Math. Soc. (2) **50**, 438–446 (1949); these Rev. **10**, 571] is either recurrent or $R^2=2R_{\alpha\beta}R^{\alpha\beta}$. If $R=0$, a recurrent V_n is symmetric; if $R_{\alpha\beta}=0$, it is locally Euclidean. (3) If V_4 is orientable with $R_{\alpha\beta}=0$ and with Euler-Poincaré characteristic less than 2, then V_4 is locally Euclidean. *C. B. Allendoerfer.*

Reeb, Georges. Sur la courbure moyenne des variétés intégrales d'une équation de Pfaff $\omega=0$. C. R. Acad. Sci. Paris **231**, 101–102 (1950).

Consider a compact orientable Riemannian manifold. To a field of hyperplane elements a scalar invariant can be defined which reduces to the mean curvature of the integral hypersurface when the field is completely integrable. The author proves that the integral of this invariant times the volume element is equal to zero. *S. Chern.*

*Finsler, Paul. **Über die Krümmungen der Kurven und Flächen.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 463–478, Rome, 1943.

This paper contains a number of ideas and results for a theory of curvatures of curves and surfaces in a Finsler space, in the same spirit as in the author's thesis [Göttingen, 1918]. The emphasis is on the geometrical approach. After a discussion of the curvature theory of curves and the angular metric, the author suggests the normal curvature, which depends also on direction, as the first curvature of a surface. If the surface is in a higher dimensional space, curvatures of higher order can be defined in a similar manner. Discussion is also given of the question of the intrinsic equations of a surface in this sense. *S. Chern.*

Moór, Arthur. Généralisation du scalaire de courbure et du scalaire principal d'un espace finslérien à n dimensions. Canadian J. Math. **2**, 307–313 (1950).

The scalar of curvature and the principal scalar defined by Berwald for two-dimensional Finsler spaces are generalized to Finsler spaces of n dimensions. *S. Chern.*

Varga, O. Über das Krümmungsmass in Finslerschen Räumen. Publ. Math. Debrecen **1**, 116–122 (1949).

Berwald has defined a curvature for a bivector in Finsler space which generalizes the Riemannian sectional curvature. On the other hand, Finsler has defined a curvature for two-dimensional Finsler space generalizing the Gaussian curvature. The author proves that the latter, for the surface formed by the extremals tangent to the bivector, is equal to the Berwald curvature, thus generalizing a well-known theorem in Riemannian geometry. Proof is by means of an osculating Riemannian metric along a congruence of curves. *S. Chern* (Chicago, Ill.).

Dufresnoy, Jacques, et Revuz, André. Introduction au calcul différentiel extérieur tensoriel. II. Bull. Tech. Univ. Istanbul **2**, 13–26 (1949). (French. Turkish summary)

[For part I, see the same Bull. **1**, 48–66 (1948); these Rev. **11**, 401.] An exposition of the theory of tensorial exterior differential calculus, both in Euclidean space and in Riemannian space. *S. Chern* (Chicago, Ill.).

Tonolo, Angelo. Sopra una classe di deformazioni finite. Ann. Mat. Pura Appl. (4) **29**, 99–114 (1949).

In a previous paper [Pont. Acad. Sci. Acta **13**, 29–53 (1949); these Rev. **11**, 461] the author has derived necessary and sufficient conditions for the principal directions of the Ricci tensor to be V_2-normal. The reviewer of that paper remarked that these conditions hold for any tensor of rank 3 with 3 different eigenvalues whether it is the Ricci tensor or not. In this paper the same conditions are derived for the tensor of a deformation. *J. A. Schouten* (Epe).

Moreau, Jean-Jacques. Tenseurs à structure isotrope. C. R. Acad. Sci. Paris **230**, 1734–1735 (1950).

An enumeration of all numerical tensors of five or less indices under the group of proper rotations in 3-space. *H. P. Robertson* (Washington, D. C.).

NUMERICAL AND GRAPHICAL METHODS

Conolly, B. W. A short table of the confluent hypergeometric function $M(\alpha, \gamma, x)$. Quart. J. Mech. Appl. Math. **3**, 236–240 (1950).

This gives $M(\alpha, \gamma, x)={}_1F_1(\alpha; \gamma; x)$ to 11 decimal places for parameters $\alpha=-1(0.2)1$, $\gamma=0.2(0.2)1$, and for arguments $x=0.1, 0.2(0.2)1$. Values were computed, using a desk machine, from the power series, the coefficients of which had already been tabulated by J. C. P. Miller. Various functional checks were applied. The results are believed to be correct within two or three units of the last place. *J. Todd* (Washington, D. C.).

*Thorne, C. J. **A table of harmonic and biharmonic polynomials and their derivatives.** Supplement to Bulletin no. **39** of the Utah Engineering Experiment Station. Salt Lake City, Utah, [1949]. ii+3+156 pp.

The first table gives, at $x=1$, the real and imaginary parts, and the partial derivatives (up to order 3) of the real part, of $(x+iy)^n/n!$ and $(x-iy)(x+iy)^{n-1}/(n-1)!$ for $n=1(1)25$, $y=0(0.05)1$, $\frac{1}{8}$, $\frac{1}{4}$. Succeeding tables give similar values for $y=0$, $x=y$, $y=\frac{1}{2}$, $x=2y$, $y=\frac{3}{4}$, $x=3y$. The tables have been computed using desk machines by the use of recurrence relations and some functional checks were carried out; ten

significant figures were kept in all calculations and the results are given to this extent. A small number of values were checked using Salzer [Table of Powers of Complex Numbers, National Bureau of Standards, Applied Mathematics Series, No. 8. United States Government Printing Office, Washington, D. C., 1950; these Rev. 11, 618] and found correct. While the tables have been prepared in view of specific applications [cf. Bulletin no. 39 of the Utah Engineering Experiment Station, Salt Lake City, Utah (1949); these Rev. 12, 64], the values of $x^n/n!$ available may be of more general use. The tables have been mimeographed and are reasonably legible; there are occasional omissions and misprints. *J. Todd* (Washington, D. C.).

van Wijngaarden, A. Table of the cumulative symmetric binomial distribution. Nederl. Akad. Wetensch., Proc. 53, 857–868 = Indagationes Math. 12, 301–312 (1950).

A table accurate to five decimal places is given for $P(n, c) = 1 - 2^{1-n}\sum_{i=0}^{c}\binom{n}{i}$ for $n \leqq 200$, $0 \leqq c \leqq n$. Since

$$P(n, c) = 1 - 2I_{\frac{1}{2}}(n-c, c+1),$$

where

$$I_x(p, q) = \int_0^x t^{p-1}(1-t)^{q-1}dt \Big/ \int_0^1 t^{p-1}(1-t)^{q-1}dt,$$

$P(n, c)$ can be obtained directly from tables of the incomplete beta-function as far as they go (i.e., up to $n = 100$). *H. Chernoff* (Urbana, Ill.).

Curry, Haskell B., and Lotkin, Max. A study of fourth order interpolation on the ENIAC. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 613, ii+23 pp. (3 plates) (1946).

Curry, Haskell B., and Wyatt, Willa A. A study of inverse interpolation on the ENIAC. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 615, 100 pp. (1946).

Lotkin, Max. Inversion on the ENIAC using osculatory interpolation. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 632, i+42 pp. (1947).

In these three papers the authors have studied in a detailed fashion the problem of carrying out both direct and inverse interpolation on the ENIAC, an electronic digital machine. These papers were written shortly after the machine was placed in operation and were clearly intended by the authors to be the first papers of a series on the use of the machine for problems of interest to the Ballistic Research Laboratories. Each of them is concerned, first, with the mathematical theory of the problem and, second, with the complete and precise coding for the ENIAC. Since they were written the control organ of the ENIAC has been completely changed [cf. the following review]. The papers have, therefore, served their initial purpose and their latter halves are of less interest now than when they were first written. *H. H. Goldstine* (Princeton, N. J.).

Clippinger, R. F. A logical coding system applied to the ENIAC. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 673, 39 pp. (1948).

The author gives in this report a careful and detailed description of how to code a problem for the ENIAC. It is intended to "make the task of coding problems so clear and straightforward that physicists, aerodynamicists, and applied mathematicians could code their own problems and prepare tests." The coding of a problem for the ENIAC in its original form was a highly difficult and complex operation and the physical realization of the code on the machine was quite arduous. It was accomplished by establishing physical wire and switch connections numbering in the thousands for each problem. In this system each problem was a completely new experience and little or no use could be made of previous coding knowledge beyond mere skill and facility. Von Neumann worked out a scheme for a priori wiring the ENIAC to "understand" and "obey" a fixed repertoire of about 60 basic orders which contain the usual arithmetic operations, certain logical instructions, and suitable transfer operations. Under this regime the coder merely writes out his code in terms familiar to the mathematician, subject to the restriction that his total vocabulary is 60 "words." These orders are then coded into decimal digits and set on manually operated switches. The ENIAC seriatim executes these orders, i.e., the appropriate fixed wire connections are actuated and the operation performed. This report describes von Neumann's system and contains enough general explanatory material to make the operation of the machine understandable to the reader. *H. H. Goldstine.*

Lynch, J., and Johnson, C. E. Programming principles for the IBM Relay Calculators. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 705, 94 pp. (1949).

During the years 1941–1944 the I.B.M. Co. in cooperation with some members of the staff of the Ballistic Research Laboratories developed a relay calculator as an aid to the computing program at the Laboratories. Two of these machines were delivered in 1944 and were placed in test operation. After this test period a number of changes were proposed to increase the capacity, versatility, and overall efficiency of these machines. These changes were made and the revised machines returned to Aberdeen in 1946. The present report is a text on these machines for the use not only of Laboratory personnel but also for interested outsiders. It gives a clear and comprehensive discussion of all phases of the relay calculators. *H. H. Goldstine.*

Doodson, A. T. A method for the smoothing of numerical tables. Quart. J. Mech. Appl. Math. 3, 217–224 (1950).

A method is described, which is partly numerical and partly graphical, for smoothing numerical tables based on observational data. *T. N. E. Greville.*

* **Reicheneder, Karl. Fehlertheorie und Ausgleichung von Rautenketten in der Nadirtriangulation.** Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam, no. 1, Akademie-Verlag, Berlin, 1949. viii+98 pp.

Salzer, Herbert E. Coefficients for polar complex interpolation. J. Math. Physics 29, 96–104 (1950).

In two previous papers, I and II, [same J. 24, 141–143 (1945); 26, 56–61 (1947); these Rev. 7, 219; 8, 535] the author has developed formulas for Lagrangian interpolation for complex analytic functions tabulated around a circular arc at points differing in phase by a constant angle θ. The present paper provides the numerical values of the coefficients in these formulas for $\theta = 1°, 5°, 10°, 15°, 20°$, and $30°$. Part I gives coefficients to eight places for the 3-, 4-, and 5-point formulas of paper I while part II gives auxiliary coefficients to nine places for the 3-, 4-, 5-, 6-, 7-, 8-, and 9-point formulas of paper II. *W. E. Milne.*

Teixidor, J. Arithmetical demonstration of an interpolation formula. Gaceta Mat. (1) 1, 139–141 (1949). (Spanish)

An elementary algebraic proof is given of the fact that Newton's interpolation formula to rth differences is exact for polynomials of degree r. The proof could be shortened by making use of the fact that such a polynomial is determined by $r+1$ values. *T. N. E. Greville.*

Rosser, J. Barkley. Note on zeros of the Hermite polynomials and weights for Gauss' mechanical quadrature formula. Proc. Amer. Math. Soc. 1, 388–389 (1950).

The main purpose of this note is to point out that, in most cases, use of the mechanical quadrature formula to compute integrals of the form $\int_{-\infty}^{\infty} e^{-x^2} f(x)dx$ is unsatisfactory. *Extract from the paper.*

de Kok, F. Numerical integration. Euclides, Groningen 25, 271–273 (1950). (Dutch)

The author gives a formula for the integral of $f(x)$ from a to b in terms of four ordinates, two at the ends and two others irrationally spaced within the interval. An expression for the error is derived. *W. E. Milne.*

Aitken, A. C. Studies in practical mathematics. V. On the iterative solution of a system of linear equations. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 52–60 (1950).

The author points out that, in general, iterative procedures for the solution of the vector equation $Ax=h$ can be described as follows. The matrix A is expressed in the form $B-C$ where B^{-1} is readily obtainable. The iteration procedure involves the relationship $x^{t+1}=B^{-1}Cx^t+B^{-1}h$ or equivalently $x^{t+1}-x^t=B^{-1}C(x^t-x^{t-1})$. The two major cases are those in which B is diagonal and the Seidelian case in which B is the diagonal plus the part of the matrix A below the diagonal. In the case in which A is positive definite, the process is shown to converge. The difference form for the iterative procedure indicates the dependence of the process on the characteristic roots of $B^{-1}C$ and, if the roots are real, the iteration process can be extrapolated after it becomes uniform. In the case in which A is positive definite, $A=D-C-C'$ where D is diagonal, C subdiagonal, C' the transpose of C, it is shown that the Seidelian process, in which in each alternate cycle the order of the variables is reversed, is governed by the matrix $(D-C')^{-1}C(D-C)^{-1}C'$ and that the roots of this matrix are real. Consequently, for a positive definite A, this process permits one to accelerate convergence by extrapolating. Numerical examples are given. *F. J. Murray* (New York, N. Y.).

Tatevskiĭ, V. M., and Finkel'steĭn, A. I. An approximate method for evaluating special determinants. Vestnik Moskov. Univ. 1948, no. 10, 63–74 (1948). (Russian)

Let the elements of the determinant $D=|a_{ij}|$ of nth order satisfy the conditions: $\underline{A}<a_{ii}<\bar{A}$; $\underline{a}<a_{kj}<\bar{a}$, $k\neq j$; $\rho=|\bar{a}/\underline{A}|<1$. Let N_{nl} denote the number of terms in the expansion of D which contain exactly l nondiagonal elements, and when all the elements are positive let N_{nl}^{+} (N_{nl}^{-}) denote the number of those terms which are positive (negative). An approximate value of D is obtained if in the expansion all terms containing more than m ($m<n$) nondiagonal elements are neglected. An estimate of the error is given in terms of ρ and N_{nl}. If all the elements of D are positive then the estimate of the error may be sharpened by using N_{nl}^{+} and N_{nl}^{-}. The authors develop formulas for N_{nl}^{+}, N_{nl}^{-}, and N_{nl} and have tabulated their values for $n=3, \cdots, 8$. Useful tables of $N_{nl}^{*}\rho^l$ for $\rho=.5, .3, .2$, and $.1$ are also included. An example determinant of order 5, for which $\underline{A}=\bar{A}=10$, $\bar{a}<1$, is evaluated, and it is shown that if only those terms containing 3 or more diagonal elements are used (11 of the 120 terms) the error is only 2%. *A. W. Goodman* (Lexington, Ky.).

Gleyzal, A. N., and Fifer, S. A general method for the numerical calculation of the roots of a polynomial equation. David W. Taylor Model Basin, Rep. no. 568, 8 pp. (1948).

A method is given for finding the roots, real and complex, of a polynomial equation $P(z)=0$ with real coefficients. Graeffe's root-squaring process is employed to find the 2^pth powers of the roots of $P(z)=0$ and a root-cubing process to find the 3^qth powers of the roots of $P(z)=0$. Since $1=2^r s+3^q t$, where s and t are integers, if α is a root of $P(z)=0$, then $\alpha=(\alpha^{2^p})^s\cdot(\alpha^{3^q})^t$. When $\beta=\alpha^{2^p}$, $\bar{\beta}=\bar{\alpha}^{2^p}$ are found from the root-squaring process and the corresponding values of $\gamma=\alpha^{3^q}$, $\bar{\gamma}=\bar{\alpha}^{3^q}$ from the root-cubing process, the two roots α and $\bar{\alpha}$ are given either by the pair $\beta^s\gamma^t$, $\bar{\beta}^s\bar{\gamma}^t$ or the pair $\beta^s\bar{\gamma}^t$, $\bar{\beta}^s\gamma^t$. Here the amplitudes of the roots determine which of the two pairs is correct. An example which illustrates the process is given. *E. Frank.*

Meulenbeld, B. Note on Lill's method of solution of numerical equations. Nederl. Akad. Wetensch., Proc. 53, 464–469 = Indagationes Math. 12, 116–121 (1950).

A graphical method is described for determining the real roots of a numerical equation. The method generalizes Lill's orthogone method. The author presents a rigorous proof and shows how the method can be used to determine graphically the derivatives of a polynomial $f(x)$. *S. Levy.*

Palm, F. W. Anwendung und Verallgemeinerung des graphischen Verfahrens von Winckler. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 157, 275–297 (1950).

L'auteur développe une méthode de construction graphique de: $y=a_0x^n+a_1x^{n-1}+\cdots+a_n$ due à Winckler [Akad. Wiss. Wien, S.-B. IIa. 53, 326–338 (1866)]. Il expose à ce sujet un certain nombre de considérations de géométrie algébrique. Il montre que cette méthode est supérieure à celles de Segner et de Lill [voir, par exemple, Willers, Practical Analysis . . . , Dover, New York, 1948, pp. 78–79; ces Rev. 10, 404] pour la détermination des différences divisées successives: $z=(y-y_1)/(x-x_1)$, $t=(z-z_2)/(x-x_2)$, etc. (Il faut toutefois remarquer que le procédé exige le tracé d'un plus grand nombre de droites que les deux procédés indiqués plus haut.) Une généralisation du procédé permet inversement de déterminer les valeurs numériques d'un polynome écrit sous la forme:

$$a_n+(x-p_1)(a_{n-1}+(x-p_2)[a_{n-2}+(x-p_3)\{\cdots\}]).$$

J. Kuntzmann (Grenoble).

Matthieu, P. Über das Extrapolationsverfahren von Adams zur angenäherten Lösung gewöhnlicher Differentialgleichungen. Elemente der Math. 5, 25–31 (1950).

Continuation of an earlier expository paper [same journal 4, 34–42 (1949); these Rev. 10, 485].

Head, J. W. The solution of Lagrangian frequency equations with complex coefficients or roots by the escalator method. The application of the method to problems connected with flutter and radio circuits. Aircraft Engrg. 22, 104–105, 108 (1950).

The escalator process for determining frequencies described elsewhere by Morris [Philos. Mag. (7) 38, 275–287

(1947); The Escalator Method in Engineering Vibration Problems, Wiley, New York, N. Y., 1947; these Rev. **9**, 210, 382] or by Morris and Head [same journal **14**, 312–314, 316 (1942); Philos. Mag. (7) **35**, 735–759 (1944); these Rev. **4**, 148; **7**, 84] is here shown to apply without radical alteration to the case where the coefficients are complex and to the case where the latent roots and associated modes are complex.

W. E. Milne (Los Angeles, Calif.).

Walters, E. R., and Rea, J. B. Determination of frequency characteristics from response to arbitrary input. J. Aeronaut. Sci. **17**, 446–452 (1950).

Essentially, the problem dealt with is that of determining the Fourier coefficients of a periodic function by numerical integration. The function is specified graphically or, presumably, in an equivalent form. Let the function, $f(t)$, have a period T. Setting $\omega_0=2\pi/T$ and dividing the period into $M+1$ equal intervals Δt, the coefficients of the nth cosine term and nth sine term are given approximately by

$$a_n=(\Delta t/T)\sum_{m=0}^{M} f(m\Delta t)\cos(n\omega_0 m\Delta t),$$

$$b_n=(\Delta t/T)\sum_{m=0}^{M} f(m\Delta t)\sin(n\omega_0 m\Delta t).$$

It is observed that each summation for any given n can be interpreted as the nth element in a column matrix. In fact, writing $f(m\Delta t)=f_m$, $\cos(n\omega_0 m\Delta t)=g_{nm}$, and $\sin(n\omega_0 m\Delta t)=h_{nm}$, then, except for the factor $\Delta t/T$, we have

$$\begin{bmatrix} g_{nm} \\ h_{nm} \end{bmatrix}\begin{bmatrix} f_m \end{bmatrix}=\begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

This seems to be a particularly convenient form for punched card computation. A machine process for matrix multiplication is described. A numerical example is given in which the transfer function of a linear dynamical system and an applied driving force are assumed. The output function of the system is then computed analytically. Starting with the assumed driving function and calculated output function as arbitrary graphs, their Fourier series are computed numerically by the method described and the transfer function computed from these data. The result obtained agrees very closely with the transfer function originally assumed.

R. Kahal (St. Louis, Mo.).

Biot, A. Transformation graphique des coordonnées colorimétriques. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 115–119 (1950).

A fairly simple rectilinear construction, made possible by the projective character of the transformation.

A. S. Householder (Oak Ridge, Tenn.).

Cagnet, Michel. Intégrateur optique permettant le calcul des répartitions des intensités lumineuses dans l'image d'un objet étendu. C. R. Acad. Sci. Paris **230**, 2084–2086 (1950).

An apparatus is described which permits the rapid calculation of the distribution of luminous flux in the image of an object produced by an optical system, provided that the distribution of flux in the image of a point source is known. The result is obtained by a photometric measurement.

W. E. K. Middleton (Ottawa, Ont.).

Cagnoli, Carlo. Il piano di costruzione di una nave in progetto ricavato da un verticale integrale ausiliario. Atti Accad. Ligure **5**, 121–134 (1949).

In connection with its use in naval architecture the author describes an integraph which he has built. It is slightly simpler in design than the widely used instruments of Abdank-Abakanowicz as produced by Coradi. The linkage is adjustable in position on a long narrow rolling frame. The tracing points are arranged so that (it is claimed) it can be used almost as well for drawing the derivative. Unlike other integraphs intended for differentiating as well [e.g., E. von Harbou, Z. Angew. Math. Mech. **10**, 563–585 (1930)] this one has no mirror or prism to assist in following a given curve with the knife edged wheel. No report of tests for accuracy are given.

R. Church (Annapolis, Md.).

Hramov, D. N. On the gravimetric deduction of the deviation of a plumb line. Akad. Nauk SSSR. Bull. Inst. Teoret. Astr. **4**, no. 3(56), 126–133 (1949). (Russian)

The deviation $\Delta\varphi$ at the origin of the plumb line from the vertical in the meridian-plane caused by the local anomaly $\Delta g(r, \alpha)$, where r and α are polar coordinates in the horizontal plane is proportional to the integral of $\Delta g(r, \alpha)\cdot r^{-2}\cos\alpha$ over the area of a circle of radius R around the origin. The author replaces this integral by three inexact expressions $\pi R(\partial(\Delta g)/\partial x)_0$, $\pi[\Delta g(R, 0)-\Delta g(R, \pi)]/2$, $\int_0^{2\pi}\Delta g(l, \alpha)\cos\alpha\, d\alpha$ and compares their values with the exact integral on the hypothesis of a homogeneous spherical disturning mass (point-source) whose center is at a depth h under the point $r=r_0<R$, $\alpha=\pi$, as well as on the hypothesis of a horizontal line-source. The comparison is illustrated graphically for different values of the ratio $R/h=\lambda$. The result is rather negative since the curves diverge for $\lambda>\frac{1}{2}$ and λ must be taken large to get a good value.

E. Kogbetliantz (New York, N. Y.).

Jecklin, Heinrich. Algebraische Begründung einer Klasse versicherungstechnischer Approximationen. Mitt. Verein. Schweiz. Versich.-Math. **50**, 133–140 (1950).

The author discusses approximations for various actuarial functions such as single premiums and premium reserves of joint life insurances. His formulae are closely related to Lidstone's formula and are based upon the following two approximations:

(1) $$a_{x_1x_2\overline{m}}a_{\overline{m}}\sim a_{x_1\overline{m}}a_{x_2\overline{m}};$$

(2) $$a_{x_1x_2\overline{m}}+a_{\overline{m}}\sim a_{x_1\overline{m}}+a_{x_2\overline{m}}.$$

E. Lukacs (Washington, D. C.).

MECHANICS

Hydrodynamics, Aerodynamics

Ertel, Hans, und Rossby, Carl-Gustav. Ein neuer Erhaltungs-Satz der Hydrodynamik. S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1949**, no. 1, 10 pp. (1949).

The authors prove a result which can be stated as follows. Given a circulation-preserving motion whose acceleration-potential is ϕ^*, i.e., $a=-\text{grad}\,\phi^*$, where a is the acceleration, define $W=\int_0^t(\frac{1}{2}v^2-\phi^*)dt$. Then

$$(D/Dt)\{\sigma w\cdot(v-\text{grad}\,W)\}=0,$$

where D/Dt is the material derivative, σ is the specific volume, w is the vorticity vector, v is the velocity vector. When the same motion is viewed by an observer in a coordinate system rotating with respect to that in which it is circulation-preserving, the authors show that a result of the same form but with a somewhat different function W continues to hold. The proofs employ the Lagrangian description, and are rather elaborate. The authors promise to publish Eulerian proofs in another paper.

C. Truesdell (Bloomington, Ind.).

Gião, Antonio. Sur les équations intégrales de l'hydrodynamique. J. Phys. Radium (8) **11**, 219–226 (1950).

The author in effect considers a differential equation of type $\partial V/\partial t=L(V,t)$, where L is a differential operator with respect to space variables. Let κ be an arbitrary parameter. Then $\partial V/\partial t-\kappa\Delta V=A$, where $A\equiv L-\kappa\Delta V$. The author takes the gradient of this equation and solves it by means of a process analogous to that of Oseen. His result expresses $V_{,i}$ at time t in a region in terms of its values at time 0 throughout the region and the values of A and V on the boundary during the entire time between 0 and t. The value of V itself can then be obtained by integration. In the application to hydrodynamics the function A contains the pressure, the instantaneous velocity field and its gradient, the extraneous forces, and the viscous forces. These are actually interrelated in a complicated way and cannot be prescribed or known upon a closed boundary at all times. Thus the author's application of his method to the equations of viscous compressible fluids must be regarded as a system of integral equations for the velocity gradient in the interior in terms of the values of itself and of other hydrodynamical quantities upon the bounding surface. The author uses his result to show that motions of perturbation of a certain type in a fluid enclosed in a vessel are determined completely by their initial values. *C. Truesdell.*

Scheid, Francis. On the asymptotic shape of the cavity behind an axially symmetric nose moving through an ideal fluid. Amer. J. Math. **72**, 485–501 (1950).

Levinson [Ann. of Math. (2) **47**, 704–730 (1946); these Rev. **8**, 419] has derived the following formula for the asymptotic shape $f(x)$ of the cavity behind an axially symmetric body moving in an ideal fluid with uniform velocity in the x direction,

$$(1)\quad f(x)=(cx^{\frac{1}{2}}/\log^{\frac{1}{4}}x)[1-\tfrac{1}{8}(\log\log x/\log x)+O\{(\log x)^{-1}\}],$$

where c is a constant. The formula is obtained from estimates of certain integrals which appear in a singular, nonlinear integro-differential equation for $f(x)$. In this paper, the author makes a reappraisal of these integrals, obtaining sharper estimates from which is derived the following strengthening of (1),

$$f(x)=(cx^{\frac{1}{2}}/\log^{\frac{1}{4}}x)[1-\tfrac{1}{8}(\log\log x/\log x)+(\tfrac{1}{4}+\tfrac{1}{2}\log(\tfrac{1}{2}c))/\log x+(5/128)(\log\log x/\log x)^2-(7/32+(5/16)\log(\tfrac{1}{2}c))(\log\log x/\log^2 x)+O\{(\log^2 x)^{-1}\}].$$

In addition to the general smoothness assumptions on $f(x)$ made by Levinson, it is assumed here that the derivative of the order of magnitude term in (1) is $O\{(x\log^2 x)^{-1}\}$.

D. Gilbarg (Bloomington, Ind.).

Eisenberg, Phillip. An approximate solution for incompressible flow about an ellipsoid near a plane wall. J. Appl. Mech. **17**, 154–158 (1950).

The incompressible potential flow past an ellipsoid near a plane wall parallel to its axis is approximated by using a disturbance potential equal to a multiple (suitably chosen) of the sum of that due to the ellipsoid and its image ellipsoid separately. The theory is compared with experiment.

M. J. Lighthill (Manchester).

Bragg, Stephen L., and Hawthorne, William R. Some exact solutions of the flow through annular cascade actuator discs. J. Aeronaut. Sci. **17**, 243–249 (1950).

The authors study the flow of an incompressible, inviscid fluid through a three-dimensional annular region, which, using cylindrical coordinates (r,θ,z), may be defined by the inequalities $0<r_h\leqq r\leqq r_t$, $-\infty<z<\infty$. The velocity components are assumed to be continuous everywhere except for the tangential component which may have a discontinuity in z at the disc $z=0$. This disc may be regarded as the limiting form of a blade system, fixed or rotating, that occurs in axial compressors and turbines. Assuming rotational symmetry (independence from θ) for the radial, tangential, and axial components c_r, c_θ, c_z of the velocity and ξ, η, ζ of the vorticity, the authors introduce a stream function ψ so that $rc_r=\partial\psi/\partial z$ and $rc_z=-\partial\psi/\partial r$, and also the function $\phi(\psi)=rc_\theta$ so that $r\xi=-\partial\phi/\partial z$ and $r\zeta=\partial\phi/\partial r$. The tangential component of the vorticity is found from the equation $\eta r=\psi_{rr}+\psi_{zz}-(1/r)\psi_r$, and the total enthalpy H from the equation $dH=r^{-2}[\phi d\phi+\eta r d\psi]$. The function ψ is assumed to have the form $\psi=\psi_\infty+\sum_{n=1}^{\infty}e^{\pm k_n z}\psi_n'$, where the k_n are positive constants and ψ_∞ and the ψ_n' are functions of r only. The case $H=$constant, $\eta r=A\psi$, is considered in detail; it is found that ψ_∞ and the ψ_n' are then expressible as linear combinations of the Bessel functions of various orders, with coefficients which, together with the k_n, may be found from the boundary conditions at infinity, along the walls, and at the disc. The case that ηr is a function just of r is also treated, in somewhat less detail.

M. Marden (Milwaukee, Wis.).

Taylor, Geoffrey. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. Proc. Roy. Soc. London. Ser. A. **201**, 192–196 (1950).

Using an analysis similar to that given by Lamb [Hydrodynamics, 6th ed., Cambridge University Press, 1932, pp. 370 ff.], "it is shown that, when two superposed fluids of different densities are accelerated in a direction perpendicular to their interface, this surface is stable or unstable according to whether the acceleration is directed from the heavier to the lighter fluid or vice versa. The relationship between the rate of development of the instability and the

length of wave-like disturbances, the acceleration and the densities is found, and similar calculations are made for the case when a sheet of liquid of uniform depth is accelerated." Part II is an experimental investigation by D. J. Lewis [same Proc. Ser. A. **202**, 81–96 (1950)].

J. V. Wehausen (Providence, R. I.).

Coombs, A. The translation of two bodies under the free surface of a heavy fluid. Proc. Cambridge Philos. Soc. **46**, 453–468 (1950).

In treating the problem of the title, the author restricts himself to the two-dimensional problem and makes the usual assumptions of an inviscid fluid and the linearized boundary conditions at the free surface. In order to illustrate his method, he first derives expressions for the lift and drag on a simple cylinder. This is specialized to the cases of a flat plate with an angle of attack and a circular cylinder, giving known results of Kočin [Proceedings of the Conference on the Theory of Wave Resistance, Central. Aero-Gidrodinam. Inst., Moscow, 1937, pp. 65–134, especially pp. 87–95] and Havelock [Proc. Roy. Soc. London. Ser. A. **157**, 526–534 (1936)]. The case of two cylinders is considered next. After a general treatment of this problem, two special cases are considered, a pair of circular cylinders with equal radii situated (1) with their centers on the same vertical plane, one 4 and the other 8 radii beneath the surface, and (2) with both centers 4 radii beneath the surface and 4 radii apart. For these cases computations of the lift and drag forces are made and are presented both in tables and graphically as functions of a Froude number.

J. V. Wehausen (Providence, R. I.).

Kočin, N. E. Theory of waves generated by oscillations of a body beneath the free surface of a heavy incompressible fluid. Učenye Zapiski Moskov. Gos. Univ. Mehanika **46**, 85–106 (1940) = Sobranie Sočinenil' [Collected Works], v. 2, pp. 277–304, Moscow-Leningrad, 1949. (Russian)

Using methods parallel to those in his earlier work on wave resistance [Proceedings of the Conference on the Theory of Wave Resistance, Central. Aero-Gidrodinam. Inst., Moscow, 1937, pp. 65–134] the author develops the subject of the title. After discussion of the formulation of the problem and of the proper boundary conditions to insure uniqueness, an expression for the velocity potential of a source of harmonically oscillating strength is derived. A distribution of such sources is then used to represent the velocity potential of an oscillating body. The integral equation for the distribution is discussed and it is shown that solutions exist at least for sufficiently small and sufficiently large values of the parameter ω^2/g, where ω is the frequency of oscillation. Expressions for the components of the force on the body, averaged over a period, are derived and expressed in terms of an analogue to the author's H-function introduced in his study of wave resistance [loc. cit.]. As examples, the author finds approximate expressions for the forces on and the free surface over (1) an expanding and contracting sphere, (2) a rigid sphere oscillating vertically, and (3) oscillating horizontally. *J. V. Wehausen.*

Papon, André. Rayon de courbure minimum au bord d'attaque d'un profil d'aile. C. R. Acad. Sci. Paris **230**, 1931–1932 (1950).

An airfoil profile in the z plane is obtained by a mapping of a curve C in the ζ plane. The transformation is assumed to possess two singular points, O and O'; C passes through O' to form the trailing edge and passes near O to form a smooth leading edge. The author finds the condition for minimum radius of curvature of the airfoil profile. He finds that it occurs at the point homologous to that at which C comes nearest to O, and he computes its value in terms of the properties of C and of the transformation near O.

W. R. Sears (Ithaca, N. Y.).

Taylor, J. Lockwood. Lift correction for finite chord. A method of close approximation that is not laborious and is applicable in general to wings without sweep. Aircraft Engrg. **22**, 174 (1950).

On the basis of the exact result for a rectangular wing having a uniform vorticity distribution in incompressible flow, the author proposes a correction of the lift curve slope of a flat wing, which appears to be superior to that furnished by lifting line theory and reasonably close to that of lifting surface theory for a wing of aspect ratio six.

J. W. Miles (Los Angeles, Calif.).

Synge, J. L. Note on the kinematics of plane viscous motion. Quart. Appl. Math. **8**, 107–108 (1950).

By using Green's theorem and the existence of a solution to the second boundary value problem of potential theory, the author obtains the following characterization theorem. In order that two given functions w and θ be such that there exist a plane continuously differentiable motion of a material filling a finite domain S, adhering to its boundary without slipping, and possessing $w=2\omega$ as its vorticity and θ as its expansion, it is necessary and sufficient that $\int_S(wU-\theta V)dA=0$, where U is an arbitrary harmonic function and V is its conjugate function. The special case $\theta=0$ was given by G. Hamel [Göttingen Nachr. Math.-Phys. Kl. **1911**, 261–270]; the author notes an error in Hamel's proof. [Reviewer's note. The author's proof of sufficiency is somewhat abbreviated, in that he assumes the existence of a solution to his system (8), viz., $u_x'+v_y'=\theta$, $v_x'-u_y'=w$, $(lu'+mv')_B=0$, where l and m are the direction cosines of the outward normal to the boundary B. Apart from questions of regularity, a solution to this system exists if and only if $\int\theta dA=0$. This requirement need not be stated separately, however, since it is the special case $V=1$, $U=0$ of the author's condition.] *C. Truesdell* (Bloomington, Ind.).

Tomotika, S., and Aoi, T. The steady flow of viscous fluid past a sphere and circular cylinder at small Reynolds numbers. Quart. J. Mech. Appl. Math. **3**, 140–161 (1950).

Using Oseen's linearization of the Navier-Stokes equations and a method of solution similar to Goldstein's [Proc. Roy. Soc. London. Ser. A. **123**, 225–235 (1929)], the authors have made detailed computations of the steady flow patterns around a sphere and a circular cylinder for small Reynolds numbers. They find that for a Reynolds number (based on radius) as small as 0.1 a stationary vortex ring exists behind the sphere and that for a Reynolds number as small as 0.05 a stationary vortex pair exists behind the cylinder. They find also that for a sphere the frictional drag is twice the pressure drag (a well-known result for the Stokes approximation) and that for the cylinder they are equal. A computation of the drag coefficient of a cylinder, based on three terms of a series expansion, is also given for several small Reynolds numbers (from 0.8 to 8).

J. V. Wehausen (Providence, R. I.).

Meksyn, D. Integration of the laminar boundary layer equation. I. Motion of an elliptic cylinder. Separation. Proc. Roy. Soc. London. Ser. A. **201**, 268–278 (1950).

The differential equation for the laminar boundary layer, in a form developed by the author, is first written into an integro-differential equation. The solution is then obtained by using a power series and by substituting the power series into this integro-differential equation. Reasons are given why the second form of the solution can be expected to give much better accuracy with a limited number of terms. The boundary condition is also more readily imposed in the second form. After these steps are taken, an ordinary differential equation is obtained which essentially governs the variation of skin friction with down-stream distance. This is to be integrated numerically to determine the point of separation. Certain general conclusions are drawn about the point of separation. In applying the method to the case of the Schubauer ellipse, it is found necessary to approach the point of separation from the down-stream side to get good prediction. The agreement is then very close.

C. C. Lin (Cambridge, Mass.).

Meksyn, D. Integration of the laminar boundary layer equation. II. Retarded flow along a semi-infinite plane. Proc. Roy. Soc. London. Ser. A. **201**, 279–283 (1950).

The method of the previous paper is applied to the case of linearly decreasing free-stream velocity with reasonably satisfactory results. *C. C. Lin* (Cambridge, Mass.).

***Kočin, N. E. The problem of the flow of a viscous fluid about a semi-infinite plate.** Sobranie Sočinenil [Collected Works], v. 2, pp. 493–504, Moscow-Leningrad, 1949. (Russian)

This is a posthumous paper dating from 1944 which was prepared for publication by A. A. Dorodnicyn. In the two-dimensional Navier-Stokes equations the pressure is eliminated, a stream function ψ is introduced and the resulting equation $\psi_x\Delta\psi_x-\psi_x\Delta\psi_y=\nu\Delta\Delta\psi$ is transformed to parabolic coordinates (ξ, η), where $\eta=0$ is the equation of the semi-infinite flat plate. The author looks for a solution in the form $\psi\sim\xi f_0(\eta)+\xi^{-1}f_1(\eta)+\xi^{-3}f_2(\eta)+\cdots$. This leads to a sequence of differential equations of the fourth order,

$$\nu f_n^{(4)}+f_0f_n'''+(2n+1)f_0'f_n''+f_0''f_n'-(2n-1)f_0'''f_n = F_n(\eta; f_0, \cdots, f_{n-1}, \cdots, f_0^{(4)}, \cdots, f_{n-1}^{(4)}).$$

The equation for f_0 gives the Blasius solution. The solutions of the equation for f_n, in a nondimensionalized form, are discussed and their general form obtained. The work was apparently not carried far enough to given any specific corrections to the Blasius formula. *J. V. Wehausen.*

Wuest, W. Beitrag zur instationären laminaren Grenzschicht an ebenen Wänden. Ing.-Arch. **17**, 193–198 (1949).

The author studies the unsteady motion over an infinite wall moving parallel to itself and near the boundary of two fluids sliding past each other. In either case, equations of the type of heat conduction are obtained, and well-known results can be used. For example, the solution can be given in a closed form when the plate motion may be described as a polynomial in time. Results of the two-fluid problem are applied to wind blowing over water surface. It is stated that in another paper the present results will be used as a first approximation in the discussion of the usual stationary boundary layer problem [see the following review].

C. C. Lin (Cambridge, Mass.).

Wuest, W. Entwicklung einer laminaren Grenzschicht hinter einer Absaugestelle. Ing.-Arch. **17**, 199–206 (1949).

The author considers the flow over two semi-infinite plates, parallel to each other, and with leading edges displaced relative to each other by a distance x_0 parallel to the plates and a distance y_0 perpendicular to them. Suction between the plates is just sufficient to remove the lower part of the boundary layer over the first plate. The boundary layer over the second plate is then investigated by using the results of the above paper after approximating the boundary layer equation in the von Mises form by the equation of heat conduction. *C. C. Lin.*

Onsager, L. Statistical hydrodynamics. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 279–287 (1949).

In the first part of this paper the author considers the two-dimensional motion of n vortex points with coordinates (x_i, y_i) and circulations k_i, $i=1, \cdots, n$. The equations of motion can be written in Hamiltonian form, $k_i dx_i/dt=\partial H/\partial y_i$, $k_i dy_i/dt=-\partial H/\partial x_i$, $i=1, \cdots, n$, where

$$H=-(2\pi)^{-1}\sum_{i<j}k_ik_j\log\{(x_i-x_j)^2+(y_i-y_j)^2\}^{\frac{1}{2}},$$

and x_i and y_i play the rôles of the conjugate coordinates in dynamics. The phase space for the vortex system may then be introduced and the motion treated in the manner of statistical mechanics. The author sketches a proof that in certain cases the stronger vortices will tend to cluster so that the dominant pattern will be that of a few very strong vortices. The limitations of the model and its connection with the theory of turbulence are discussed. The second part of the paper is a brief exposition of the main ideas of the Kolmogorov-Heisenberg-Onsager statistical theory of turbulence. *J. V. Wehausen* (Providence, R. I.).

Frenkiel, François N. Étude statistique de la turbulence. Fonctions spectrales et coefficients de corrélation. O.N.E.R.A. Rap. Tech. no. **34**, 103 pp. (1 plate) (1948).

[This paper was completed in 1942, but its publication was delayed.] In the first chapter the author emphasizes the difference between time- and space-correlations and spectra of turbulence as well as between the Lagrangian and Eulerian viewpoint. He gives some new relations for isotropic and homogeneous turbulence and discusses the validity of the relation between Taylor's time-spectrum and the longitudinal space-correlation. After having initiated the use of one-dimensional space-spectra, the author gives the relations between the longitudinal and transverse spectra. The discussion of three-dimensional spectra (introduced by the reviewer and other authors after the completion of this paper) is not included. In the following chapters the author shows how experimental correlations and spectra can be represented by empirical curves. Numerous functions are discussed and the conditions of their use are discussed. Von Karman's equation is used to find analytically the transverse correlation when the longitudinal correlation function is determined and, reciprocally, the inverse equation given by the author is used to compute analytically the longitudinal correlation curves from the empirical transverse correlation functions. The spectral functions are computed analytically for the various correlation functions and examples of spectral and correlation curves are given in numerous curves.

J. Kampé de Fériet (College Park, Md.).

Chandrasekhar, S. On Heisenberg's elementary theory of turbulence. Proc. Roy. Soc. London. Ser. A. **200**, 20–33 (1949).

The author works out the spectrum of decaying isotropic turbulence on the basis of similarity of the spectrum and Heisenberg's assumption for the transfer function (which appears to be the best so far suggested). Numerical results are completely worked out for various values of the Reynolds number, revealing several important features. For the stationary spectrum at high frequencies an explicit solution can be obtained [see the author, Physical Rev. (2) **75**, 896–897 (1949); these Rev. **10**, 493]. *C. C. Lin.*

Chandrasekhar, S. On Heisenberg's elementary theory of turbulence. Symposium on turbulence, 1 July 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1136**, pp. 43–48 (1950).

An abbreviated version of the paper reviewed above. *C. C. Lin* (Cambridge, Mass.).

Rotta, J. Das Spektrum isotroper Turbulenz im statistischen Gleichgewicht. Ing.-Arch. **18**, 60–76 (1950).

The author considers the power laws of decay and the spectrum of turbulence following Heisenberg's theory. The one-dimensional spectrum thus calculated (with turbulence energy varying inversely as time) is compared with the measurements of Simmons and Salter with satisfactory agreement [cf. the 2d preceding review and reference cited there]. *C. C. Lin* (Cambridge, Mass.).

Bers, Lipman. On the continuation of a potential gas flow across the sonic line. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2058**, 58 pp. (1950).

This paper shows a number of sufficient conditions for analytic continuation across the sonic line of a two-dimensional transonic flow. If the velocity components and their first derivatives are continuous on the sonic line and if the velocity vector is neither parallel nor perpendicular to the sonic line, then there exists a unique solution in the neighborhood of the sonic line both in the physical and in the hodograph plane, corresponding in one-to-one manner. No general statement concerning the extent of the supersonic region is made. A Cauchy problem is formulated in the hodograph plane by approximating the coefficient of the Chaplygin differential equation with a group of step functions. *C.-C. Chang* (Baltimore, Md.).

Rey Pastor, Julio. On the linearized equation of supersonic flow. Ann. Mat. Pura Appl. (4) **30**, 91–96 (1949). (Spanish)

The author considers that the methods hitherto adopted for discussing the velocity potential which satisfies the linearized equation of supersonic flow are unsatisfactory. If $(\xi, \eta, 0)$ is a point on the plane $z=0$ of a nearly flat wing and within the Mach cone of $P(x, y, z)$, the velocity potential at P is of the form

$$\varphi=\int\int \lambda(\xi, \eta)[(x-\xi)^2-(y-\eta)^2-z^2]^{-\frac{1}{2}}d\xi d\eta$$

taken over the region within the Mach cone of P. This integral is convergent, but differentiation under the integral sign is not applicable. The author adopts the method of singular integrals [E. W. Hobson, Theory of functions . . . , vol. 2, 2d ed., Cambridge University Press, 1926, chapter 7]. In particular he discusses the case of a rectangular wing symmetrical with respect to the plane $z=0$ at zero incidence, span $2l$, chord a, Mach number $2^{\frac{1}{2}}$, $l>a$, and shows rigorously the existence of an unperturbed region behind the trailing edge. The treatment is summary and a full development is to appear later. *L. M. Milne-Thomson.*

Pillow, A. F. The formation and growth of shock waves in the one-dimensional motion of a gas. Proc. Cambridge Philos. Soc. **45**, 558–586 (1949).

The author examines the process of shock formation and propagation in a one-dimensional flow of a perfect gas. The compression wave is initiated by a piston moving into the still gas. The apparatus of characteristic theory is invoked to provide a solution in the portions of the (x, t)-plane prior to first appearance of the shock. The shock, however, introduces a floating boundary across which the Rankine-Hugoniot relations must be satisfied. To avoid the consequences of this difficulty the author makes two assumptions. First, the piston is assumed to move with constant acceleration. This insures that the shock forms at, rather than behind, the initial compressive disturbance, and permits the use of certain closed expressions in the formulation of boundary conditions. Second, it is assumed that the new impulses reaching the shock from behind are there wholly absorbed. These assumptions lead to a first approximation for the equation of the shock and of the flow field near it. This first approximation is then made to yield a second, for which the mathematical expression is a second-order linear hyperbolic differential equation. A Riemann function is found in terms of hypergeometric functions, and the first few terms of the power series solution are given. The solution remains valid until the shock attains sufficient strength to invalidate the no-reflection condition initially postulated.

D. P. Ling (Murray Hill, N. J.).

Rand, Robert C. Prandtl-Meyer flow behind a curved shock wave. J. Math. Physics **29**, 124–132 (1950).

Let P be a symmetrical convex profile obtained by constructing through the vertices O and O' of an isosceles triangle OVO' half lines normal to OO'. Consider a symmetrical steady nonviscous completely supersonic plane flow with attached shock wave produced by a uniform isentropic stream impinging on P. In accordance with a common practice, the nonisentropic flow in the entire region behind the shock has been approximated by wedge flow near VO and Prandtl-Meyer expansion about O. To test the accuracy of this approximation, it has been fitted through a shock wave to a nonisentropic shear flow which is uniform near the axis of symmetry. Computations of velocity and stagnation pressure show that for sharp profiles and moderate Mach numbers the shear flow is very nearly uniform and isentropic. *J. H. Giese* (Havre de Grace, Md.).

Noyes, Robert N. Prandtl-Meyer flow for a diatomic gas of variable specific heat. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2125**, 22 pp. (1950).

The Prandtl-Meyer corner flow of a diatomic gas with variable specific heats is compared with the classical solution in which the specific heats are constant. Such variations in specific heats are appreciable when high fluid temperature and high expansion pressure ratios are involved simultaneously. The variation in specific heats has an appreciable effect on the flow but a close approximation can be obtained by using a properly chosen constant value for the ratio of specific heats. *M. H. Martin.*

Jones, Robert T. Leading-edge singularities in thin-airfoil theory. J. Aeronaut. Sci. **17**, 307–310 (1950).

"If the thin-airfoil theory is applied to an airfoil having a rounded leading edge, a certain error will arise in the determination of the pressure distribution around the nose. It is shown that the evaluation of the drag of such a blunt-nosed airfoil by the thin-airfoil theory requires the addition of a "leading-edge force," analogous to the well-known leading-edge thrust of the lifting airfoil. The method of calculation is illustrated by application to (1) the Joukowski airfoil in subsonic flow and (2) the thin elliptic cone in supersonic flow. The paper concludes with a general formula for the edge force which is applicable to a variety of wing forms." From the author's summary. *J. W. Miles.*

Gardner, C. A relation between time-dependent and steady linearized supersonic flows past conical bodies. Comm. Pure Appl. Math. **3**, 39–43 (1950).

The linearized calculation of the disturbance in a supersonic flow due to the nonstationary motion of a wing having a supersonic leading edge and a single subsonic side edge is reduced to a pair of steady flow problems as follows:

$$(1)\quad \varphi_{xx}-\varphi_{yy}-\varphi_{zz}-\varphi_{tt}=0,\quad (\varphi_z)_{z=0\pm}=g(x,y,t),\quad (\varphi)_{x<0}=0,$$

$$(2)\quad \chi_{xx}-\chi_{zz}-\chi_{tt}=0,\quad (\chi)_{t=0}=g,\quad (\chi)_{x<0}=0,$$

$$(3)\quad \psi_{tt}-\psi_{yy}-\psi_{zz}=0,\quad (\psi_z)_{z=0}=\chi,\quad (\psi)_{t<-x}=0,$$

$$(4)\quad \varphi(x,y,z,t)=[\psi(\xi,x,y,z,t)]_{\xi=0},$$

where φ is the velocity potential for the original problem (1), (2) and (3) are the auxiliary problems, and t is a linear combination of x and the true time. The author produces an explicit solution for φ for arbitrary g and states that the result may be extended to any planform that can be treated, in steady flow, by Evvard's method [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1382 (1947); these Rev. **8**, 610] and that has subsonic edges consisting of polygonal lines. No applications of the method are given, and it appears that the required integrations, even in the case of the rectangular wing, the solution for which is known, present a formidable problem. The reviewer notes that the lower limit of integration in (25) should read $(t_1^2+x^2)^{\frac{1}{2}}$ and the lower limit in (26) $-(\eta^2+z^2)^{\frac{1}{2}}$. *J. W. Miles* (Los Angeles, Calif.).

Gardner, C. Time-dependent linearized supersonic flow past planar wings. Comm. Pure Appl. Math. **3**, 33–38 (1950).

Following a procedure analogous to that in the paper reviewed above, the author reduces the problem of solving the (complete) linearized potential equation for a time dependent supersonic flow, subject to a boundary condition on the surface of a cone, to an equivalent steady flow problem. With respect to the mathematical treatment of the problem proposed by the author, the reviewer remarks that the solution may not be regarded as complete until the boundary condition at infinity (Sommerfeld's radiation condition) is included. However, there is the even stronger criticism that the consistent linearization of the time dependent problem, with the fineness ratio, Mach number, and reduced frequency as characteristic parameters, yields Laplace's equation (in planes transverse to the flow) in the neighborhood of the body, rather than the more complete linearized equation appropriate to the planar problem. [See G. N. Ward's treatment of the steady flow problem, Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); these Rev. **10**, 644.] Hence, the author's procedure would involve rather more labor than could be justified by the final accuracy of his results, as intrinsically limited by the initial assumptions. *J. W. Miles* (Los Angeles, Calif.).

Tomotika, S., and Hasimoto, Z. On the transonic flow of a compressible fluid through an axially symmetrical nozzle. J. Math. Physics **29**, 105–117 (1950).

By a method similar to that developed by S. Tomotika and K. Tamada [Quart. Appl. Math. **7**, 381–397 (1950); these Rev. **11**, 275] for computing two-dimensional mixed isentropic flows in the sonic region, the flow in the vicinity of the throat of an axially symmetrical nozzle is studied. Several exact solutions to von Kármán's equation for axially symmetrical transonic flows are obtained and the one that gives flows through a converging and diverging nozzle is considered in detail. This solution consists of four branches, of which two are rejected because of singularities. Of the remaining two branches, one gives pure supersonic flow and the other gives Taylor's type of flow with a local supersonic region in the throat. By varying a parameter, the latter branch approaches two asymptotes which yield Meyer's type of asymmetrical flows. *Y. H. Kuo.*

Kramers, H. A. On the behaviour of a gas near a wall. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 297–304 (1949).

The slip velocity u_0 of gas at the wall when there is a shear velocity gradient du/dz normal to the wall was computed approximately by J. C. Maxwell [Philos. Trans. Roy. Soc. London. Ser. A. **170**, 231–256 (1879)] assuming (1) that the molecular velocity distribution function at the wall for incident molecules is not influenced by the re-emitted molecules from the wall, and (2) that there is a fraction f of the impinging molecules re-emitted perfectly diffuse, whereas the rest $1-f$ undergo specular reflection. For isothermal flow, the slip number k, defined in terms of the mean free path l as $u_0=kldu/dz$, is then $(2-f)/f$. Experimental results give $k\sim 1.2$. Thus $f<1$. The present author believes that the most reasonable assumption is to take $f=1$, and the deviation of k from unity must be sought from a more exact solution of the molecular velocity distribution function near the wall. Starting from the Boltzmann integro-differential equation for the molecular velocity distribution function, the author deduces a linear homogeneous integral equation for the deviation from the Maxwellian distribution function, and k is then an eigenvalue of the problem. It is evident that k must be dependent upon the manner of interaction between molecules. No solution is given. However the author shows that k must be larger than unity by qualitative arguments. *H. S. Tsien.*

Åström, E. Magneto-hydrodynamic waves in a plasma. Nature **165**, 1019–1020 (1950).

Herlofson, N. Magneto-hydrodynamic waves in a compressible fluid conductor. Nature **165**, 1020–1021 (1950).

Topoliansky, D. Sur le mouvement des eaux souterraines dans un milieu non homogène en présence de sources de drainage. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 91–96 (1947). (Ukrainian. Russian and French summaries)

On envisage ici un problème spécial concernant le mouvement des eaux souterraines dans un milieu à deux couches en présence d'une source de drainage dans le cas du déversoir sans palplanches. *From the author's summary.*

Elasticity

Kauderer, H. Über ein nichtlineares Elastizitätsgesetz. Ing.-Arch. 17, 450–480 (1949).

The linearity of classical elasticity is closely connected with the assumption that the strain is small, and the most natural way to pass to nonlinearity is to assume the strain finite and to take for its components the well-known expressions involving quadratic terms. In this way we arrive at the general theory of Murnaghan [Amer. J. Math. 59, 235–260 (1937); Proc. Symposia Appl. Math., Vol. I, pp. 158–174 (1949); Anais Acad. Brasil. Ci. 21, 329–336 (1949); these Rev. 11, 67, 627]. The author proceeds quite differently and without reference to Murnaghan. He assumes the strain small and takes for it the classical linear expressions $(\epsilon_x=\partial u/\partial x, \cdots, \psi_{xy}=\partial u/\partial y+\partial v/\partial x, \cdots)$. On the basis of four postulates he arrives at the following formulae for stress in terms of strain: $\sigma_0=3K\kappa(\epsilon_0)\epsilon_0$, $T'=2G\gamma(\psi_0^2)D'$. Here σ_0, ϵ_0 are the arithmetic means of the three principal stresses and strains; $T'=T-\sigma_0 E$, $D'=D-\epsilon_0 E$, where T, D are the stress and strain matrices and E the unit matrix; K, G are elastic moduli (constants); ψ_0 is a strain invariant, $\psi_0^2=\frac{2}{3}[2\sum(\epsilon_x-\epsilon_0)^2+\sum\psi_{xy}^2]$; $\kappa(\epsilon_0)$, $\gamma(\psi_0^2)$ are functions, arbitrary except that each tends to unity as its argument tends to zero, so as to give Hooke's law in the limit. For these functions power series are used, and it is here that one would expect nonlinearity to collapse into linearity, the retention of higher-order terms appearing unjustified since ϵ_0 is already inaccurate because of the assumed linear form for strain. But this is not the case if the coefficients in the power series become compensatingly large, just as in $1+k\epsilon_0$ the second term may be comparable to the first if a large k balances a small ϵ_0. Thus the author produces a nonlinear elasticity with small (linear) strain by using large subsidiary elastic constants. Simple exact solutions of special problems are given, without specialising the functions κ and γ; the bending and torsion of a prism are also discussed. The theory is kept in contact with engineering by the introduction of numerical values. *J. L. Synge* (Dublin).

***Milne-Thomson, L. M. Finite elastic deformations.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 33–40. Eng. library has.

Using dyadic notation, the author develops formulae for finite strain. By a thermodynamic argument [not wholly clear to the reviewer] he establishes the existence of a strain energy density u for isothermal and for isentropic conditions. He defines an elastic deformation as one for which u exists as a continuous one-valued function of the Lagrangian strain tensor alone (i.e., the tensor D_0 such that $d\mathbf{P}\cdot d\mathbf{P}-d\mathbf{P}_0\cdot d\mathbf{P}_0=2d\mathbf{P}_0\cdot D_0\cdot d\mathbf{P}_0$, where $\mathbf{P}_0$, $\mathbf{P}$ are initial and final positions respectively). For such deformations he expresses stress in terms of strain. An isotropic material is then defined as one in which u is a function of the Eulerian strain tensor alone (i.e., the tensor D such that $d\mathbf{P}\cdot d\mathbf{P}-d\mathbf{P}_0\cdot d\mathbf{P}_0=2d\mathbf{P}\cdot D\cdot d\mathbf{P}$). The author refers to Murnaghan [Amer. J. Math. 59, 235–260 (1937)], but does not make it plain in what respects his paper differs from Murnaghan's apart from notation. [For Murnaghan's later matrix treatment, see Proc. Symposia Appl. Math., Vol. I, pp. 158–174 (1949); Anais Acad. Brasil. Ci. 21, 329–336 (1949); these Rev. 11, 67, 627.] *L. J. Synge* (Dublin).

Kondo, Kazuo. The geometry of the perfect tension field. I. J. Jap. Soc. Appl. Mech. 3, 36–39 (1950).

The author considers stress fields having at each point only one nonvanishing principal component, so that they are completely represented by a single field of stress vectors. He finds necessary and sufficient conditions that such stress fields satisfy Cauchy's statical equations for a Riemannian continuum subject to no extraneous force. The resulting conditions are that the field of stress vectors be solenoidal and the stress trajectories be geodesics. Consequently, in the case of Euclidean space, the stress trajectories are straight lines. *C. Truesdell* (Bloomington, Ind.).

Pastori, Maria. Velocità di avanzamento in un mezzo anisotropo ed invarianti del tensore elastico. Ann. Mat. Pura Appl. (4) 30, 241–248 (1949).

This paper deals with invariants formed from the elasticity tensor of an anisotropic medium and their interpretation in terms of velocities of propagation, as already discussed in a previous paper by the same author [Nuovo Cimento (9) 6, 187–193 (1949); these Rev. 11, 71]. *J. L. Synge* (Dublin).

Ghizzetti, Aldo. Sugli stati di tensione piana in un corpo elastico. Ann. Mat. Pura Appl. (4) 29, 125–130 (1949).

The author obtains a new and simple reduction of the problem of plane elastic stress: $t_{iz}=0$, $i=1, 2, 3$, and t_{ij} independent of z for $i, j=1, 2, 3$. By use of integrability conditions he shows that it is necessary and sufficient for a solution that the extraneous force be solenoidal and independent of z and that

$$u=\alpha z^2-\alpha x^2+2\sigma^{-1}\beta xy-\sigma^{-1}\alpha y^2+\sigma^{-1}(1-\sigma)\gamma x+2E^{-1}(1+\sigma)\partial\Phi/\partial y-\phi,$$

$$v=\beta z^2-\sigma^{-1}\beta x^2+2\sigma^{-1}\alpha xy-\beta y^2+\sigma^{-1}(1-\sigma)\gamma y-2E^{-1}(1+\sigma)\partial\Phi/\partial x+\psi,$$

$$w=-2z(\alpha x+\beta y+\gamma),$$

$$t_{xx}=\sigma^{-1}E(2\beta y+\gamma)+2\partial^2\Phi/\partial x\partial y-(1+\sigma)^{-1}E\partial\phi/\partial x,$$

$$t_{yy}=\sigma^{-1}E(2\alpha x+\gamma)-2\partial^2\Phi/\partial x\partial y+(1+\sigma)^{-1}E\partial\phi/\partial x,$$

$$t_{xy}=\partial^2\Phi/\partial x^2-\partial^2\Phi/\partial y^2-(1+\sigma)^{-1}E\partial\phi/\partial y,$$

where E is Young's modulus, σ is Poisson's ratio, α, β, and γ are arbitrary constants, Φ is a function determined by the force field alone, and $\phi+i\psi$ is an analytic function of $x+iy$. These results constitute a notable simplification over the usual treatment in that instead of Airy's biharmonic stress function only an ordinary analytic function $\phi+i\psi$ is required. The author indicates a class of boundary problems which are reducible to the Neumann problem and another reducible to the Dirichlet problem. *C. Truesdell.*

Wells, A. A. On the plane stress-distribution in an infinite plate with a rim-stiffened elliptical opening. Quart. J. Mech. Appl. Math. 3, 23–31 (1950).

The author derives a solution for the decrease of stress concentration factor at an elliptical opening in a plate under uniform plane stress at infinity when a compact stiffening rim is applied to the opening. Basing his method on that already used for a circular opening, he writes down equations for the boundary conditions at the periphery of the opening, which have not been explicitly given in any of the references. Similar equations are then written down for the conditions at the periphery of the elliptic opening; from the point of view of the reviewer it would have been more satisfactory if the physical conditions which give rise to these equations had been more explicitly stated. Thereafter, using known solutions for the plate with a plain elliptical opening under the given conditions at infinity, and which automatically satisfy the boundary conditions between the rim and plate, an additional stress distribution is super-

imposed, which satisfies the given conditions between the rim and plate and reduces to zero at infinity. The author then derives expressions for the stresses in the plate and the rim for the particular cases of hydrostatic tension and pure shear in two directions at infinity, noting that from these all other cases may be solved. These results depend on a parameter defined in terms of the normal cross section, thickness and curvature of the rim and certain conclusions are drawn with regard to the maximum values of the stresses and their relation to the distribution of thickness and cross section of the rim. *R. M. Morris* (Cardiff).

Guzman, Arturo M., and Luisoni, Cesar J. New solutions for some particular loadings of plane plates. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 196, Serie segunda, 18, Revista 4, 268–287 (1949). (Spanish. English summary)

Galerkin's approximate method for finding the deflection of a thin plate subject to a normal distributed load q (i.e., for finding a real-valued function $w(x, y)$ satisfying the partial differential equation

$$\partial^4 w/\partial x^4+2\partial^4 w/\partial x^2\partial y^2+\partial^4 w/\partial y^4=\Delta\Delta w=q/N$$

in a plane domain D, where q is a given real-valued function defined on D, N is a real constant, and w is subject to certain homogeneous boundary conditions), starting with n functions $\varphi_1, \cdots, \varphi_n$ which satisfy the homogeneous boundary conditions, consists in determining n real numbers $a_1, \cdots, a_n$ from the system of linear equations

$$(*) \qquad \sum_{i=1}^{n} a_i \int_D \varphi_j \Delta\Delta \varphi_i \, dx\,dy = \int_D N^{-1} q \varphi_j \, dx\,dy,$$

$j=1, \cdots, n$, and then taking the function $\sum_{i=1}^{n} a_i \varphi_i$ as an approximation to the deflection w. The authors refer to (*) as Galerkin's orthogonality condition. By an argument involving the principle of virtual displacements the authors replace (*) by another condition which may be applied when both concentrated and distributed loads occur. Using their condition they treat numerically the wholly simply supported and the wholly clamped rectangular plate with a concentrated load at the center, computing both displacements and moments, and compare their results with those of Timoshenko [Theory of Plates and Shells, McGraw-Hill, New York-London, 1940], Marcus [Die Theorie Elastischer Gewebe . . . , 2d ed., Springer, Berlin, 1932], and Stiles [J. Appl. Mech. 14, A-55–A-62 (1947); these Rev. 8, 421]. The simply supported rectangular plate with partial rectangular load (i.e., q a nonzero constant on a smaller concentric rectangle and zero outside it) is also dealt with numerically. In each case it is stated that the present method, from the standpoint of numerical work, is to be preferred to the previous ones. *J. B. Diaz.*

***Thorne, C. J. Symmetrically loaded rectangular plates fixed at points.** Bulletin of the University of Utah, Vol. 39, no. 10. Bulletin no. 39 of the Utah Engineering Experiment Station. Salt Lake City, Utah, 1949. ii+10+14+21+25 pp.

A solution is given for the deflections and moments in square, 2:1 and 3:1 rectangular plates clamped on the edges and subjected to a uniformly distributed lateral load in one case, and to a concentrated lateral load at the center in another. The method of solution approximates the clamped boundary condition by requiring zero slope and zero deflection at several points on the boundary. The results are in excellent agreement with those obtained by earlier authors. A supplement to this bulletin [cf. these Rev. 12, 54] gives tables used in the computations. *S. Levy.*

Tarabasov, N. D. Strength calculations for annular force-fitted plates. Doklady Akad. Nauk SSSR (N.S.) 70, 977–980 (1950). (Russian)

A thin circular plate is made by force-fitting several annular sections possessing identical elastic properties. The elastic deformation due to force-fitting is taken into account. Musheli̇švili's formulation of the plane stress problem is then used to compute the stresses in the annular sections. *H. I. Ansoff* (Santa Monica, Calif.).

Aquaro, Giovanni. Sul calcolo delle deformazioni di uno strato sferico elastico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7 (1949), 289–297 (1950).

A complete system of particular solutions of the classical equations of equilibrium in terms of displacements is established. It is then applied to the solution of the second boundary value problem (given surface tractions) for the spherical shell. The coefficients of the resulting series are obtained by solving systems of linear algebraic equations of order six. *W. Leutert* (College Park, Md.).

Šatašvili, S. H. On steady vibrations of an elastic body with given surface displacements. Doklady Akad. Nauk SSSR (N.S.) 71, 249–252 (1950). (Russian)

The solution of the problem of steady vibrations in a continuous finite elastic body with prescribed surface displacements is reduced to the solution of a system of simultaneous integral equations of the Fredholm type. A proof of the existence of a solution is briefly sketched. *H. I. Ansoff* (Santa Monica, Calif.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Schlegel, Richard. Optical refraction and Fermat's principle at a point. J. Opt. Soc. Amer. 40, 244–245 (1950).

The Weierstrass-Erdmann corner condition states that if $\int f(x, y, y')dx$ is stationary when calculated along a curve having a corner, then $\partial f/\partial y'$ is continuous across the corner. The author uses this result to prove Snell's law of refraction; for if $f=v^{-1}(1+y'^2)^{\frac{1}{2}}$, where v is the velocity of light, then $y'/\{v(1+y'^2)^{\frac{1}{2}}\}$ is continuous across the corner. It is also claimed that the corner condition implies that the optical path is straight when v is constant. This seems to be wrong; for surely all that the condition implies is that the actual ray can have no corners if v is constant. *E. T. Copson.*

Zimmermann, Fritz. Drehstromunsymmetrieprobleme in Matrizendarstellung. Arch. Elektrotechnik 38, 131–140 (1944).

The analysis by Kirchhoff's laws of a three phase circuit when the circuit is not symmetrical (as a result of unbalanced loads, unbalanced faults, or short circuits that are not symmetrical in the three phases) becomes extremely complicated and difficult. Symmetrical components is the method now generally employed for calculating such circuits. It was first given by C. L. Fortescue [Trans. Amer. Inst. Elec. Engrs. 37, 1027–1140 (1918)].

In this paper the concise notation of matrix algebra is employed in finding the expression for the symmetrical

components of the currents in a star connected three phase network under a very general set of unbalanced conditions. The transformation matrices and their inverses by means of which phase currents and voltages, and passive impedances may be expressed in terms of their symmetrical components, and vice versa, are given. Assuming that the phase voltages and impedances are known, one proceeds with the solution for the phase currents by obtaining first the matrices of the symmetrical components of the voltages and impedances. Applying the Kirchhoff law, the matrix of the symmetrical components of the currents is obtained as the product of the reciprocal impedance matrix times the voltage matrix. The general result, when all possible dissymmetries such as might occur in the generator, transmission lines, transformers, etc., as well as the load itself are accounted for, is a rather complicated algebraic expression. The author discusses the results to be obtained with some useful special cases, such as systems symmetrical up to the load, systems in which one, two, or three lines are grounded, and systems in which there occurs a short circuit across two lines.

R. Kahal (St. Louis, Mo.).

Shmoys, Jerry. Diffraction of electromagnetic waves by a plane wire grating. New York University, Washington Square College, Mathematics Research Group, Research Rep. no. **EM-18**, i+25 pp. (1950).

"Previous work on diffraction by a wire grating is briefly reviewed. The problem of diffraction by a finite grating consisting of perfectly conducting wires of arbitrary cross-section is formulated in terms of characteristic plane waves corresponding to the various order spectra defined in optics. Scattering matrix elements are expressed as stationary functionals of current distribution on the grating wires, for the incident wave falling at right angles to the grating elements and polarized either parallel or perpendicular to them. Similar expressions are obtained for the impedance matrix elements; these are evaluated for the thin wire grating, and an equivalent network constructed." From the author's summary.

A. Heins (Pittsburgh, Pa.).

Hetrick, David L. Propagation of the TM_{01} mode in a metal tube containing an imperfect dielectric. J. Appl. Phys. **21**, 561–564 (1950).

"The TM_{01} mode in a circular dielectric-filled wave guide, where dielectric losses and metal wall losses are considered simultaneously, is analyzed from the viewpoint of the boundary value problem. General expressions for the attenuation and phase constant are obtained, and the percentage errors introduced by (1) neglect of wall losses and (2) assumption that the total attenuation is the sum of dielectric attenuation and approximate wall loss attenuation are computed for a particular case." From the author's summary.

A. Heins (Pittsburgh, Pa.).

Graffi, Dario. Sulla propagazione delle onde elettromagnetiche in un tubo conduttore riempito da un dielettrico eterogeneo. Ann. Mat. Pura Appl. (4) **30**, 233–239 (1949).

The author discusses the propagation of $TE_{m,0}$-waves (i.e., three nonvanishing field components, E_y, H_x, and H_z) in a rectangular waveguide filled with a nonhomogeneous dielectric. It is assumed that the permeability is constant, while the dielectric constant is a function of one transverse coordinate, $\epsilon=\epsilon(x)$, in the interior of the guide, $0<x<a$. If ϵ_1 and ϵ_2 are respectively the minimum and maximum values of ϵ, the lowest critical frequency lies between the limits $(2a(\epsilon_2\mu)^{\frac{1}{2}})^{-1}\leqq\nu_c\leqq(2a(\epsilon_1\mu)^{\frac{1}{2}})^{-1}$. For a frequency $\nu>n(2a(\epsilon_1\mu)^{\frac{1}{2}})^{-1}$ there are at least n waves in the tube, each propagated with a definite phase velocity V_r, such that

$$(\epsilon_2\mu-(r^2/4a^2\nu^2))^{-\frac{1}{2}}\leqq V_r\leqq(\epsilon_1\mu-(r^2/4a^2\nu^2))^{-\frac{1}{2}},\quad r=0,1,\cdots,n,$$

and having a corresponding phase constant $\beta_r=\omega/V_r$. The group velocities, V_r', defined by $1/V_r'=\partial\beta_r/\partial\omega$, may be obtained from the simple relationship $V_rV_r'=1/\epsilon_m\mu$, where ϵ_m is some mean value of ϵ. Since $V_r>(\epsilon_2\mu)^{-\frac{1}{2}}$ it follows that $V_r'<(\epsilon_2/\epsilon_1)^{\frac{1}{2}}(\epsilon_1\mu)^{-\frac{1}{2}}$, where $(\epsilon_2\mu)^{-\frac{1}{2}}$ and $(\epsilon_1\mu)^{-\frac{1}{2}}$ are respectively the minimum and maximum velocities of free waves in the dielectric medium.

M. C. Gray (Murray Hill, N. J.).

Foà, Emanuele. Su alcune proprietà dei mezzi eterogenei mediamente omogenei. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **5** (1947/48), 141–147 (1949).

The author first formulates a slight generalization of Kelvin's minimum theorem for steady currents: if ρ is the resistivity, j the current density vector, and E' the impressed e.m.f., then the actual current field j yields a smaller value for $\int(\rho j^2-2E'\cdot j)dV$ than does any other solenoidal field. He then obtains estimates for the conductivity by Rayleigh's method. He considers a heterogeneous medium in which the heterogeneity is sufficiently regular that one may regard the conductivity as having a constant value k_i in each of a finite number of small cylinders with parallel generators and with ends lying in a set of parallel planes; further, the relative volume v_i of that part of the material where k_i is the conductivity is constant throughout the whole body. By considering these small prisms as connected first in series, then in parallel, the author by appeal to the minimum principle obtains the result

$$\sum_{i=1}^{n}v_ik_i>k>\left(\sum_{i=1}^{n}v_ik_i^{-1}\right)^{-1}.$$

He then indicates that analogous results hold in steady heat flow, electrostatics, magnetostatics in the absence of electric currents, and in filtration theory. Finally he indicates that in a conducting sheet whose thickness is of the order of the dimensions of the portions of which the medium is composed we should expect k to be near its upper bound, while in a conducting wire whose diameter is of such magnitude k is near to its lower bound.

C. Truesdell.

Tihonov, A. N. On the establishment of an electric current in an inhomogeneous stratified medium. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 199–222 (1950). (Russian)

The establishment of an electric current in a horizontal conducting layer of total thickness l and variable conductivity $\sigma=\sigma(z)$, where $\sigma(z)$ is a step-function of depth z, $l\leqq z\leqq 0$, is studied in this paper, assuming a nonconducting basement below the conductor [$\sigma(z)=0$ for $z>l$] and a dipole on the OX-axis at the origin O as the source of current. Maxwell's equations are solved and the solution is applied to the particular case of a layer of constant conductivity. For this case curves are computed and drawn representing the current at a given point on the OX-axis and on the OY-axis as a function of the time t, if the distance of the given point from the origin is four times the thickness l of the layer. The curve for $E_x(y_0, t)$ with its characteristic shape seems to be very convenient for the computation of the thickness of the conducting layer, if the establishment of the current at this and other points is registered.

E. Kogbetliantz (New York, N. Y.).

Fradkin, E. S. Concerning the reaction of the self-field of a charged particle. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **20**, 211–217 (1950). (Russian)

The author investigates the effect of the reaction of its own field on the motion of a particle regarded as having a rigid charge distribution. This question has been investigated by others [e.g., M. Markov, Acad. Sci. USSR. J. Phys. **10**, 159–166 (1946); these Rev. **8**, 183], some of whom [e.g., Bohm and Weinstein, Physical Rev. (2) **74**, 1789–1798 (1948)] concluded that such a particle was capable of undergoing translatory oscillatory motion under the influence of its own field. In the present work the electromagnetic potentials due to the charge of the particle are determined from the field equations by the use of Fourier integrals, and from them the self-force of the particle is determined. The calculation is essentially the same as that of Bohm and Weinstein except for the fact that the time t_0 which occurs in the solution for the vector potential (and for which the potential and its time derivative are prescribed) is taken to be finite by the author while it was taken to be $-\infty$ by Bohm and Weinstein. In the absence of an external force, the author sets up the nonrelativistic equations of motion and shows that for small velocities no periodic solutions, corresponding to self-oscillations, exist. According to him, the periodic solutions of Bohm and Weinstein arise from the choice $t_0=-\infty$ which, however, is incorrect for periodic motion. Taking several different charge distributions (or form factors) and considering small velocities, the author eliminates the self-force and obtains equations of motion in a form involving the acceleration, the external force, and a finite number of their time derivatives. *N. Rosen.*

Kneschke, Alfred. Die Elektronenbewegung in magnetischen Wechselfeldern. Arch. Elektr. Übertragung **4**, 165–172 (1950).

The classical Lorentz equations for the orbit of an electron in a magnetic field of the form $H_0+\bar{H}\cos\omega t$ are integrated and some properties of the solutions are discussed.

A. H. Taub (Urbana, Ill.).

Quantum Mechanics

***Iñiguez Almech, Jose M.ª. Quantum mechanics.** Mem. Acad. Ci. Zaragoza (2) **2**, xii+522 pp. (1949). (Spanish)

This book gives a very full exposition of the well established parts of quantum mechanics in a style, and on a level, which seem well adapted to the needs of graduate students of physics. The following list of the titles (freely translated) of the nineteen chapters will suffice to give a general idea of the contents: Equations of Hamilton and Jacobi; equations of relativistic mechanics; quantum mechanics; metric spaces; linear operators; eigenvalues of a linear operator; waves and matter; Schrödinger's theory; general equations of (quantum) mechanics; various problems concerning the motion of a particle; movement of a particle in a central field of force; hydrogen atom; approximate methods for the determination of eigenfunctions; atoms with several electrons; Dirac's equation; molecules; groups; group theory in quantum mechanics; quantum theory of valence.

It will be clear from the above that the contents of the book present no essential novelties. Except for occasional minor lapses, the exposition of the mathematical theory is clear, and as complete and rigorous as can be expected, considering the class of readers for whom the work is intended. So far as the reviewer is able to judge, the discussions of the physical phenomena are satisfactory also.

However, one adverse criticism must be made. In common with many other writers of expository books on this subject, the author takes the verbal expressions "wave mechanics," "wave function," etc. too seriously; and accordingly he bases a considerable part of the discussion in chapter VII and elsewhere upon the ordinary hyperbolic wave equation (with the derivative $\partial^2\psi/\partial t^2$), regardless of the fact that the true Schrödinger "wave equation" is, formally speaking, parabolic (with the derivative $\partial\psi/\partial t$). This procedure forces him into some rather confusing maneuvres when he comes to setting up the definitive equations in chapter VIII. Such defects in their books would be avoided if writers on quantum mechanics would courageously bring themselves to set forth their full axiomatic apparatus in the very early chapters. *L. A. MacColl* (New York, N. Y.).

Symonds, N. Particle equations in generalized co-ordinates. Proc. Roy. Irish Acad. Sect. A. **53**, 115–131 (1950).

The equations proposed by Bhabha [Rev. Modern Physics **17**, 200–216 (1945); these Rev. **7**, 272] for the wave functions of a particle of any half-integral or integral spin are generalized to apply to any coordinate system in a Riemannian space. The method used is that previously used by other authors for the Dirac equations. *A. H. Taub.*

***Chang, T. S. Remarks on the theory of S matrices.** National Peking University Semi-Centennial Volume, Mathematical, Physical and Biological Series, pp. 81–88, 1948.

The author gives a relativistic definition of the S (scattering) matrix by defining it in terms of a wave matrix Ψ. The latter relates the wave function of the system with interaction to the wave function of a similar system consisting of noninteracting incident particles. A relativistic transformation law for the wave matrix is proposed and consequences of it are studied. It is shown that Ψ cannot be an invariant. *A. H. Taub* (Urbana, Ill.).

Chang, T. S. The quantum mechanics of localizable dynamical systems. Physical Rev. (2) **78**, 592–596 (1950).

This paper contains a review of Dirac's theory of localizable dynamical systems [Physical Rev. (2) **73**, 1092–1103 (1948); these Rev. **10**, 225] and obtains conditions for the expectation values of dynamical variables or their densities to be independent of the parametrization of the surface. It also discusses the relation between Dirac's theory and the earlier one of Weiss [Proc. Roy. Soc. London. Ser. A. **156**, 192–220 (1936)]. *A. H. Taub* (Urbana, Ill.).

Chang, T. S. On Weiss's theory of fields. Chinese J. Phys. **7**, 265–277 (1949). (English. Chinese summary)

The integrability of the covariant Schrödinger equation, expressed in terms of the change of the wave functional on a space-like surface caused by a change in the surface, is first shown for a theory derived from a Lagrangian which contains only first derivatives. The proof consists of the demonstration that the total change in the wave functional due to two infinitesimal changes in the surface is independent of the order in which the changes are made. The argument is then extended to theories derived from a Lagrangian which contains an arbitrary but finite number of derivatives of the field variables. Finally, the explicit forms of the energy-momentum tensor and of the angular momentum

tensor are discussed for the case that the space-like surfaces are planes. *R. Karplus* (Cambridge, Mass.).

Schweber, S. A note on commutators in quantized field theories. Physical Rev. (2) **78**, 613–614 (1950).

Recently E. P. Wigner [same Rev. (2) **77**, 711–712 (1950); these Rev. **11**, 706] investigated the extent to which the commutation relation between a pair of conjugate variables in quantum mechanics is determined by the condition that these variables obey the classical equations of motion. In the present work Schweber investigates the same question for the variables describing a quantized field. In the case of a scalar field he finds that the specification of the Hamiltonian and the field equations and the requirement of Lorentz invariance do not determine uniquely the commutation relations among the field variables. However, this lack of uniqueness does not permit one to obtain a high-frequency cut-off, as can be done in the nonrelativistic case.

N. Rosen (Chapel Hill, N. C.).

Blohincev, D. I. Field theory of extended particles. Amer. Math. Soc. Translation no. **19**, 17 pp. (1950).

Translated from Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 566–574 (1948); these Rev. **10**, 345.

Úlehla, Ivan. A contribution to the theory of self energy of the electron. Časopis Pěst. Mat. Fys. **75**, 89–95 (1950). (Czech. English summary)

The author attempts to avoid the ambiguities inherent in calculations based on the use of a not properly normalizable state functional of the electromagnetic field [Belinfante, Physical Rev. (2) **76**, 226–233 (1949); these Rev. **11**, 146]. He recommends as a rule for the calculations to introduce a cut-off not only in the momentum space ($|\vec{k}| \leqq K$) but also in the space of the photon numbers ($n_3, n_4 \leqq x$) and to normalize the state functions in this bounded space region. The limits $x \to \infty$, $K \to \infty$ are not to be taken until all other operations (calculation of the matrix elements, summation over the various intermediate states) are performed. In this way one obtains just the desired value of the self energy. For another rule see Belinfante [Physical Rev. (2) **78**, 642 (1950)]. *M. Brdička* (Prague).

Katayama, Y., and Takagi, S. Note on five-dimensional space and the self-energy of electron. Progress. Theoret. Physics **5**, 336–338 (1950).

Electrodynamics is formulated covariantly in a five-dimensional space. The masses of electron and photon are not determined by the theory but are inserted explicitly at the end of a calculation. As an example, the self-energy of the electron is computed and is found to vanish. This result can be understood if one notes that in the five-dimensional theory the masses of the particles in virtual states do not have a fixed value but must be averaged in a certain way. This averaging is just equivalent to Pauli-Villars regularization. *R. Karplus* (Cambridge, Mass.).

Sawada, Katuro. Note on the self-energy and self-stress. II. Progress. Theoret. Physics **5**, 236–251 (1950).

The radiative self-energies and self-stresses discussed in an earlier paper [same vol., 117–132 (1950); these Rev. **11**, 633] are recalculated with the use of the covariant perturbation theory of Feynman. The self-stress of the fermion due to its interaction with scalar, pseudoscalar, or vector mesons is finite and independent of the meson mass because the self-energy diverges only logarithmically. The self-stress of bosons due to fermions diverges; it can be expressed as the surface area of an infinite sphere in three-dimensional momentum space. The divergent quantities are then investigated with the aid of several cut-off procedures. For extended particles, a finite self-energy does not ensure a stable particle. Realistic and semi-formalistic regularization, on the other hand, result in a vanishing self-stress if the self-energy is made finite. (Semi-formalistic regularization consists of treating the subtraction fields realistically in intermediate states in spite of their exclusion from real states and their nonphysical properties.) *R. Karplus.*

Kinoshita, T., and Nambu, Y. On the electromagnetic properties of mesons. Progress. Theoret. Physics **5**, 307–310 (1950).

The second-order radiative corrections to the current density of integral-spin mesons interacting with the electromagnetic field are calculated covariantly by a relativistic extension of the Duffin-Kummer formalism. The results consist of extensive formulae that give corrections to the current and to the magnetic moment. Renormalization terms are separated. Spin-zero mesons, of course, have no anomalous magnetic moment, while spin-one mesons have a logarithmically divergent anomalous magnetic moment as well as an electric quadruple moment. *R. Karplus.*

Fierz, M. Zusammenhang der nicht-lokalen Felder H. Yukawa's mit solchen, die Teilchen mit dem Spin f beschreiben. Helvetica Phys. Acta **23**, 412–416 (1950).

In his paper on nonlocal fields [Physical Rev. (2) **77**, 219–226 (1950); these Rev. **11**, 567], Yukawa discussed a scalar field described by the function $U(x_i, r_i)$ and interpreted x_i as conventional space-time coordinates whereas r_i described an internal motion of an elementary particle of spin 0 whose finite extension could not be ignored. Fierz shows that U may be resolved into a superposition of particles of arbitrary spin with r_i appearing as a spin variable. In this interpretation, no suggestion arises that the particle has finite extension. The paper concludes with a significant correction of the equation (4.2) for the commutation relations in field theory in his paper on particles of arbitrary spin [same Acta **12**, 3–37 (1939)].

A. J. Coleman (Toronto, Ont.).

Gupta, Suraj N. Theory of longitudinal photons in quantum electrodynamics. Proc. Phys. Soc. Sect. A. **63**, 681–691 (1950).

The radiation field is quantized by introducing four types of photons, two transverse, one longitudinal, and one scalar, with the help of a time-like unit vector. The treatment of the scalar photons requires the use of an indefinite metric because the relevant commutator has a sign opposite to the normal one. The energy of these photons is then still positive, but states that correspond to the presence of an odd number of scalar photons occur with a negative normalization. The supplementary condition which is chosen excludes longitudinal and scalar photons from observation but is weaker than the usual one:

$$[\partial A_\mu/\partial x_\mu]^+\Phi = 0, \quad \Phi^+[\partial A_\mu/\partial x_\mu]^- = 0,$$

where $[\ ]^+$ and $[\ ]^-$ denote the positive and negative frequency parts, respectively. The permitted states of the noninteracting photon field comprise the state which contains only transverse photons and those states which contain a superposition of different numbers of scalar and longitudinal photons as prescribed by the supplementary condition.

For all of the latter, however, the normalization $\Phi^+\Phi$ vanishes; they give rise to no observable effects and can therefore be omitted from the consideration of a pure radiation field.

If the interaction of radiation with electrons is treated by a time-dependent perturbation procedure, the initial and final states are interaction free and so contain only transverse photons. It is shown, further, that the indefinite metric does not affect the perturbation formulae. This fact leads to an expression for the perturbation operator which can be employed in the usual way. It provides, of course, for the existence of nonphysical photons in virtual states. A derivation of the interaction between two electrons, for instance, leads to Møller's result; the part contributed by the longitudinal and scalar photons is just the Coulomb interaction. The Lorentz-invariant electron self-energy is also derived. All results of physical interest are independent of the time-like unit vector. *R. Karplus.*

Enatsu, Hiroshi. On the nuclear forces. Progress Theoret. Physics **5**, 102–116 (1950).

Nuclear potentials are derived with Schoenberg's [Physical Rev. (2) **60**, 468 (1941)] generalization of the Lagrange function which distinguishes spin 0 and spin 1 fields only. A canonical transformation proposed by Hove [Physical Rev. (2) **75**, 1519–1523 (1949)] is used to determine the second order approximation. In both the "symmetrical" and the "charged" theory these potentials have the same character as the potentials obtained by Bethe [Physical Rev. (2) **57**, 260–272, 390–413 (1940)] in his "neutral vector" theory ("symmetrical" theory: interaction equally strong with neutral, positive, and negative mesons; "charged" theory: interaction with charged mesons only, etc.).

E. Gora (Providence, R. I.).

Gião, Antonio. Sur la quantification du champ métrique et les interactions particules-champs. III. Systèmes de particules. C. R. Acad. Sci. Paris **230**, 1740–1742 (1950).

Gião, Antonio. Sur la quantification du champ métrique et les interactions particules-champs. IV. Application au spectre de l'hydrogène. C. R. Acad. Sci. Paris **230**, 1838–1840 (1950).

For parts I and II see the same vol., 278–280, 434–436 (1950); these Rev. **11**, 302.

Statistical Mechanics

Hinčin, A. Ya. Statistical mechanics as a problem of the theory of probability. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 3–46 (1950). (Russian)

Expository paper, based on the author's book, Mathematical Principles of Statistical Mechanics [OGIZ, Moscow-Leningrad, 1943; translated as Mathematical Foundations of Statistical Mechanics, Dover, New York, 1949; these Rev. **8**, 187; **10**, 666]. *J. L. Doob* (Urbana, Ill.).

van Hove, L. Sur l'intégrale de configuration pour les systèmes de particules à une dimension. Physica **16**, 137–143 (1950).

A one-dimensional system of particles of finite radius interacting by means of a force of finite range is considered. The Gibbs free energy is expressed in terms of the iterated kernels of a certain integral equation. By a change of variables the equation is brought into a form in which the Fredholm theory is applicable. The free energy is related to the radius of convergence of a resolvent series. It is then possible to draw conclusions as to its analytic behavior. From these it is seen that no phase transitions can occur for such a system. *K. M. Case* (Ann Arbor, Mich.).

Prigogine, I. Sur la perturbation de la distribution de Maxwell par des réactions chimiques. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 289–296 (1949).

Using a modification of the usual Chapman-Enskog method for calculating the velocity distribution function of the molecules of a gas, the author investigates the correction to the Maxwell-Boltzmann distribution which arises from the components of the gas reacting chemically with each other. Two different forms of the cross-section for the interaction of the molecules are investigated. The corrections to the rate of reaction which results from the perturbation of the Maxwell-Boltzmann distribution are written as functions of ϵ/kT (assumed greater than 1), where ϵ is the activation energy per molecule of the reaction. All these corrections vanish for $\epsilon \gg kT$ as would be expected. When $\epsilon < kT$ the method of approximation breaks down.

J. M. Luttinger (Madison, Wis.).

Domb, C. Order-disorder statistics. II. A two-dimensional model. Proc. Roy. Soc. London. Ser. A. **199**, 199–221 (1 plate) (1949).

This is a discussion of the 2-dimensional Ising model of a ferromagnet. The mathematical problem is to find the partition function

$$Z = \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_N=\pm 1} \exp\{-J\sum \sigma_i\sigma_j\},$$

where the summation in the exponential extends over all of all lattice points i and j which are nearest neighbors in the ferromagnet, and J is the interaction energy between a pair of nearest neighbors divided by kT. The calculation of Z is equivalent to the determination of the largest characteristic value of a certain matrix. Although Onsager [Physical Rev. (2) **65**, 117–119 (1944); these Rev. **5**, 280] and Kaufmann [Physical Rev. (2) **76**, 1232–1243 (1949)] have solved the characteristic value problem for a two-dimensional lattice by the use of spinor analysis, it has not yet been possible to extend this technique to a three-dimensional lattice or to a ferromagnet in an external magnetic field.

In this paper the largest characteristic value of the fundamental matrix is found as a power series in $\exp(-J)$ for a two-dimensional lattice with and without an external field by a perturbation procedure [see also Ashkin and Lamb, Physical Rev. (2) **64**, 159–178 (1943)]. In the absence of a magnetic field the terms computed by the author are in complete agreement with Onsager's exact ones. It seems that the method can be generalized for the three-dimensional lattice. The author's magnetization curve has a vertical tangent at the transition temperature. Finally the results are applied to the theory of binary solid solutions; the solubility curve for the two substances is formally closely related to the spontaneous magnetization of an Ising ferromagnet. *E. Montroll* (College Park, Md.).